LECTURE 6

DIDACTIC ENGINEERING 1. THE NOTION OF FUNDAMENTAL SITUATION.

1. DEFINITION OF THE METHODOLOGY OF DIDACTIC ENGINEERING

Suppose you want to conceive and try a new way of teaching a piece of mathematical knowledge. Then you have, roughly speaking, two ways of going about that.

Choice 1: The Comparative Study of Experimental and Control Groups

You write down a scenario of classroom activities, with a precise description of the role and actions of the teacher and the expected reponses of the students. The scenario contains advice for the teacher in case the students commit errors and mistakes of various types. The decisions made in writing the scenario pertaining to the choice of the mathematical activities and problems could be justified by curriculum prescriptions, some theory of learning, some principles of teaching, knowledge of mathematics and personal teaching experience. But the evaluation of the teaching project will not be evaluated of the basis of this justification. In fact, this justification may not be even written down or otherwise made explicit in the final report of the project. The project will be evaluated by testing the scenario on a group of students. A control group will also be chosen. The control group will be taught the same mathematical content with traditional methods, and both groups will be administered identical pre-tests and post-tests. In the case of similar results on the pre-test, if the experimental group performs better on the post-test, then the teaching project will be evaluated as ‘effective’.

Choice 2: The methodology of instructional development

You reject the ‘comparative study’ methodology because you do not believe that it is possible to teach ‘the same mathematical content’ with two different sets of mathematical activities and diferent pedagogical approaches. You also don’t believe that one can assess what the students have learned by counting their scores on a standardized test. You start by writing a detailed scenario, just like someone who picked Choice 1, but you do make explicit the rationale behind all your decisions, based on your theory of learning, your instructional theory and your theory of what it means to know the particular mathematical content that you plan to teach the students, because this is going to be the ground with respect to which your project will be evaluated. You make predictions concerning the knowledge that the students should construct as a result of participating in the planned activities. Then you try out your scenario in a class with someone else, not yourself, as a teacher. You are sitting in the classroom as an observer. You collect all possible documentation concerning the students' mathematical work. You audio- and videotape the classes, and you collect all the students' written work. Then you analyze this material with the question: Has the anticipated knowledge developed in the students? If not, then what knowledge has developed? What are the reasons behind the discrepancies between the anticipation and the actual outcome? Can these be explained in terms of the theoretical frameworks assumed a priori? Is there a need to search for an alternative theory, or for an amendment of the theory that has been used? How can the scenario be improved to decrease the difference between the anticipated and the actual knowledge produced by the scenario? On the basis of this analysis, you re-design your scenario and try it again in the same way. Etc.

If you made Choice 2 and your theoretical framework is based on the Theory of Didactic Situations, then you can say that you are using the methodology of Didactic Engineering in developing a teaching project in mathematics.

The term 'engineering' in the name of the methodology comes from an analogy of this kind of conception, design, and implementation work in mathematics education with the work of a civil engineer. An engineer does not validate his or her design of, say, a bridge, on the basis of a comparison with already constructed bridges (which have not fallen down), but on the basis of (a) predictions of its properties (stability, capacity, etc.) which can only be deduced from the mathematical and physical theory, and (b) the realization of the project, checking if the predictions were correct[1].

2. A TOOL FOR DIDACTIC ENGINEERING: THE NOTION OF FUNDAMENTAL SITUATION

If you plan to teach students some piece of mathematical knowledge so that they learn it, then the theory of situations suggests that you have to organize the didactic milieu and the game of the students with this milieu in such as way that this particular mathematical knowledge will appear as the best means available for the understanding of the rules of the game and elaborating a winning strategy (see Week 1 Class notes). You know from the theory, that there are many ways

[1] An interesting analysis of the methodology of didactic engineering can be found in the article Artigue, M. & Perrin-Glorian, M.-J., 1991: Didactic Engineering, Research and Development Tool: some Theoretical Problems linked to this Duality. *For the Learning of Mathematics 11*.1, 13-18.

in which you may fail to reach this goal (Jourdain, Topaze, Dienes effects, and other phenomena), so you have to organize the milieu so as to avoid falling into the trap of wanting to preserve the fiction of 'doing your job' at all costs.

This assumption is based on a view of mathematics, advanced by the theory of situations, according to which '*mathematical knowledge cannot be apprehended otherwise than through the activities that they allow us to realize and therefore the problems that it makes it possible to solve. Mathematics is not simply a logically consistent conceptual system for the production of rigorous proofs; it is, first of all, an activity which is realized in a situation and against a milieu. The activity is a structured one, with distinguishable phases of action, formulation and validation, devolution and institutionalization'*[2].

Theory of situations and this particular conception of mathematical knowledge can be used both to identify what mathematical knowledge is being constructed by the teacher and his or her students in an actual lesson and to 'engineer' situations aimed at the construction of a certain knowledge by the teacher and the students.

In this class we are interested in the latter activity, and our questions is: How do we go about finding out what kind of situation would generate a given mathematical knowledge?

We have to start by analyzing the knowledge K that we aim to teach. We will want to define this knowledge by the general characteristics of a *problem-situation* to which it could be considered as an optimal solution strategy. A problem-situation is more than just a problem (and it is not a school exercise); it is characterized by what is at stake in solving or not solving it, the possible states of the system it which it has appeared, rules of action, aims of solving the problem. By listing those factors (variables) of the problem-situation which are pertinent from the point of view of the knowledge K associated with it, one obtains a model called *The Fundamental Situation* associated with K.

Symbolically, $FS(K)=[C_1, \dots , C_n]$.

Normally, a piece of knowledge does not appear in its full generality, but in a variety of meanings or concepts depending on the domain of its use or application. We can assume that each such 'conception' of K is associated with certain ranges of values of the variables C_i. Thus, the FS(K) can be thought of as a generator of specific situations (SS) associated with the different conceptions of K. We have to decide which conception of K we want the students to construct.

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[2] Translated from the article
Bosch, M. & Chevallard, Y., La sensibilité de l'activité mathématique aux ostensifs. Objet d'étude et problématique. *Recherches en Didactique des Mathématiques 19*.1, 77-123, p. 81.

Example of an analysis of a piece of knowledge in terms of variables of situations in which it appears as an optimal solution - Part 1: FS and SS

Knowledge K = *Number*
FS(K) := [size of set (small, big; finite, infinite), order type of set (continuous/discrete), context of use (comparison of size of sets (which can be close to each other or far away), counting, measuring, coding elements in a set, marking rythms (chanting)), representation (numerals (in various systems: decimal, binary, roman, etc.; verbal, written), sets of physical objects, abaci, ...)]

Conception K_1 of K = everyday names of natural numbers in their *counting function*
SS-K_1 := [size of set (not very small, say 20-30; finite), order type of set (discrete), context of use (comparison of size of sets (which are far away), counting), representation (oral numerals)

The next step is to *conceive of a didactic situation* in which the teacher would devolve to the students a problem-situation whose constraints would satisfy the values of the variables of the chosen Specific Situation. This didactic situation would be defined by some values of its pedagogical variables that would ensure the devolution of the problem situation and would not contradict the problem-situation.

Example - Part 2: variables of a didactic situation

DS(K1) := [motivation (K1 must be a means to solve the students' problem, not a school problem); prerequisite knowledge (the students know the names of numbers in proper order: they can chant, 'one, two, three,…' till at least 20); kind of problem (the comparison of two biggish sets of objects placed far apart is involved in the problem to be devolved to the students)]

Only after this preliminary analysis of the knowledge to be taught can one start thinking of a possible *realization of the didactic situation in the classroom*. There are, a priori, many ways in which such a situation could be materialized, theoretically conserving the property of leading to the construction of the same knowledge by the teacher and the students[3].

Example - Part 3: a classroom realization of a SS for the counting function of number

The teacher in a kindergarten class of 5 years old children has arranged the following:
In opposite corners of a large room there are two tables. On one table there are 23 pots of paint, on the other, there are about 30 brushes. The teacher says: 'Anybody who brings, from that table over there, as many brushes as there are pots of paint on this table - I mean one brush for each pot of paint - wins a prize'. Children are motivated: they want to win a prize. They are five year olds

[3] Think of the various classroom situations proposed by the class in Week 5 with respect to the ratio conception of division.

who know their number sequence till at least 30, so the prerequisites are there. There are too many pots of paint for the children to grasp their number visually - so they will have to use some coding. They will not be allowed to use marks on paper or beads or anything other than their voice to code the number. The most economical solution will be to count the pots of paint, remember the last numeral and then count the brushes till that numeral and take those counted.

Exercise 1

(a) Define the FS for the knowledge: operation of division (FS(Div))

(b) Define the SS for the physical ratio conception of division (SS-Div/r)

(c) Define a didactic situation for SS-Div/r

Hints for a solution

(a)

The Fundamental Situation for the operation of division could be defined by the following variables:

- level of generality (an operation in abstract algebraic structures understood as the multiplication by the inverse, an operation in a concrete number structure or measure space)
- kind of entities involved (numbers only, a number and another entity (e.g. a physical magnitude, a transformation, a matrix, etc.), non-numbers only)
- kind of numbers involved, if at all (natural numbers, integers, rational numbers, real numbers, complex numbers, etc.)
- kind of representation of numbers used, if numbers used (exact, decimal approximations, decimal system, another position system)
- size of numbers, if numbers used (big, small, less/greater than 1)
- relative size of the dividend and the divisor, if numbers used (in a/b, $a < b$ or $a > b$)
- context of application (sharing, partitioning, geometric ratio, physical ratio, multiplying/combining by/with the inverse of an element in an algebraic structure)

(b) A Specific Situation related to the Physical Ratio conception of division could be defined by the following values of the above variables:

SS-Div/r is concerned with an operation in a concrete measure space, involving two physical magnitudes, expressed by rational but not integer numbers, represented in either a fractional or decimal form, and involved in a problem of comparison of ratios (otherwise the ratio would not have to be understood as a number).

Exercise 2

(a) Read Brousseau's description and analysis of a classroom situation: 'The thickness of a sheet of paper' (pp. 195-212)
(b) Determine the mathematical knowledge K and its specific conception K1 that this situation is aiming at and describe K and K1 in terms of their defining variables
(c) Describe the didactic variables of the classroom situation corresponding to K1.
(d) In what way the classroom situation realizes SS-K1?
(e) By what kind of behavior does the teacher avoid (or not) the trap of the phenomena of Topaze, Jourdain, the metacognitive shift, the metamathematical shift, the implicit suggestion of analogy?

For a better 'feel' of the above situation, I propose the following small activity in the class:
I'll put five books on the table: 1. 'Thinking mathematically', 2. 'Elementary Linear Algebra', 3. Sheakespeare, 4. Matematyka-7, 5. Robert Dictionary. I'll ask you to evaluate the thickness of the paper used in each of the books and order the books from the one with the least thick paper to the one with the thickest paper. All you will have is a ruler. How will you go about it?

When I tried to evaluate the thicknesses myself, I was measuring the thickness of all inside pages of the book and counted the number of pages. Here are the results I got:

	width	# sheets	ratio
1	12 mm	120	0.1
2	20 mm	271	0.074
3	13 mm	120	0.108
4	21 mm	231	0.909
5	57 mm	525	0.108

When I then calculated the ratios width/# sheets, it turned out that the book (5) has thicker paper than (1) which was obviously not true; I could feel it. Then I decided to measure the width of the same number of pages in each book; I chose 240 pages or 120 sheets. I then obtained the following table:

Book	width	# of sheets
1	12 mm	120

2	8 mm	120
3	13 mm	120
4	11 mm	120
5	7 mm	120

Now I did not have to calculate the ratios, I knew that the order of the books from the book with the most thin pages to that with the thickest pages is: 5, 2, 4, 1, 3.

It has become clear for me in this exercise that had I chosen to take the same number of pages from each book to start with, I would not have to calculate the ratios. So if the aim of the situation has something to do with ratios, division or rational numbers, then assigning an exercise like this one for each individual child to do on his or her own, would not help achieving it. Now I understood why the situation as described in Brousseau's book was a lot more complicated, with children not just measuring and calculating individually but in groups and having to communicate and compare their results. It is rather unlikely that all groups would have chosen the same number of sheets to measure.