1999 MATH 645 CONCORDIA UNIVERSITY

LECTURE NOTES ON THE THEORY OF DIDACTIC SITUATIONS – ANNA SIERPINSKA

LECTURE 4

THE TOPAZE, THE JOURDAIN AND THE DIENES EFFECTS IN THE PROCESS OF MATHEMATICS TEACHING

PART I - PROBLEMS OF TEACHING DIVISION - CONTINUED

In analyzing the two sets of division problems in Lecture 3, we found the following conceptions of division:

Elementary arithmetic conceptions:

Sharing: A whole number N of objects is shared among a whole number of persons P (or is put in boxes, in piles, etc.). N / P represents the number of objects that each person gets (there might be a remainder).
Partitioning: A quantity Q is distributed in packages of q; Q / q represents the number of packages. (Q and q can be any numbers)

A geometric conception:

Ratio as a comparison of two magnitudes of the same kind: if A and B are amounts of two magnitudes of the same kind then one can sometimes express the relation between these amounts as a proportion, 'A is to B as p is to q', where p and q are whole numbers, and write the ratio as a fraction p/q, without necessarily understanding it as a number in the same sense as whole numbers are numbers; in this conception, a ratio could still be understood as a pair of whole numbers.

A physical conception:

Ratio as a compound magnitude: if A and B are amounts of two magnitudes, not necessarily of the same kind, then the ratio of A to B expresses the amount of the first magnitude per a unit of the second magnitude.

An advanced arithmetic conception:

Ratio as pure number: in the conception of ratio as a compound magnitude, the ratio of the amounts A to B is abstracted from the magnitudes, and treated as a pure number.

An algebraic conception:

Multiplication by the inverse: $a/b = a*b^{-1}$.

These conceptions form a hierarchical list in the sense that every next conception (except perhaps the first two) can be construed on the basis of the previous ones, through generalization, abstraction, and extension of the meaning of terms. But every next conception has to be construed 'against' the previous ones because it requires a change of perspective, a certain reorganization of knowledge and even a rejection of beliefs related to the previous ways of understanding division.

In both textbooks, division was defined algebraically as the multiplication by the inverse but the students were not given a chance to construct this conception on the basis of and against the previous conceptions. The word problems given to the students could all be solved either with the elementary arithmetic conceptions or were so formulated that the use of the division operation was almost explicitly suggested. Having been given the procedure(s) for performing division the students were able to solve the problems without having to develop the geometric and physical notions of division. But without these conceptions, they had no grounds for developing a meaning for the procedures, and therefore could only learn the procedures but not the algebraic conception of division. The huge gap between the arithmetic conceptions of division and the algebraic conception has not been filled by an appropriate choice of problems and didactic contracts.

Everything would be fine if the authors of the textbooks said: our aim is only to teach the students some algorithms of dividing ordinary and decimal fractions or rational numbers. However, if asked, they might claim that they teach the students the notion of division, and support their claims by arguments such as:

(1) Aren't the students choosing the operation spontaneously in solving problems when appropriate?

(2) Don't the students actually correctly perform the operation of division?

Our answer would be: Yes, the students were choosing division in solving problems but this operation was strongly suggested by the text or the context of these problems[1]; the hint was

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[1] Most problems in the 1936 textbook were included in sections whose titles explicitly mentioned division, like 'Problems using division of fractions' (p. 71), and many had verbal clues within the text of the problem, like 'He *divided* the nuts into 50 equal shares', in Problem # 3, or 'what *fraction* of the full recipe…' in Problem #8 , or 'what was the *cost per mile*…' in Problem # 12.

so strong that the students did not have to choose among other possibilities - they knew, by virtue of the implicit contract, that division was what they were expected to do. This phenomenon of *giving the answer in the question* has been called by Brousseau: 'the Topaze effect' (p. 25). Concerning argument (2), what can be called 'performing the operation of division' by a mathematician, is not that at all from the point of the student, who is just, for example, changing the sign of "/" to "*", inverting the divisor, and using the multiplication table. This second phenomenon, of *giving a scientific name to a trivial activity* has been called 'the Jourdain effect' (p. 25-26).

Exercise

Find a didactic situation through which the student would be led to construct one of the ratio conceptions of division without being coerced into using this concept by an explicit hint from the teacher.

Here is what I think might work with students (but I would need to experiment it to verify my conjecture) with respect to the 'physical conception of ratio'. I would give the students a project titled: 'How much of your own weight do you eat?' Students could investigate this problem concerning themselves, members of their family, their pets (situation of action). They would come back with some data to class, and present the data to the class, perhaps in the form of tables. The problem would then arise of comparing, who eats more than someone else, not in an absolute way, but relative to one's weight (situation of formulation). For example, children may find out that a cat weighing 5 kg eats 200g of food daily and a dog weighing 20 kg eats 1 kg of food daily. Ways of making such comparisons would then be proposed and defended by different students or groups of students (situation of validation). For example, it could be proposed to compare the 'food weight per body weight' quantity. In the case of comparing the absorption of food by the cat and the dog, the question would be reduced to comparing the ratios 200 g to 5000 g and 1 kg to 20 kg. The main problem would then be: how can one compare such ratios? Maybe some students would propose to represent the ratios by fractions: 200/5000 and 1/20, and use the techniques of comparing fractions. If not, then the teacher could suggest this and negotiate the sense of using this representation with the students (institutionalization). Once the students are convinced about this representation, the teacher would pose a question in which the weights themselves would be represented by fractions of some whole and then the ratio would be a ratio of fractions, thus leading to considering the division of fractions as a ratio. For example, the teacher could propose to compare the food weight/body weight ratio of a chickadee and a sparrow, saying that a chickadee eats 9/10 of a small bird feeder daily and weighs 15/16 of the weight of the feeder, while for the sparrow the ratio is 21/25. Who eats more of his own weight and how much more? The students would then be left on their own to figure out the answer (action), then formulate their strategies (formulation), and compare their strategies for validity (validation). The teacher would then ask the students if it would be possible for them to generalize their results and say how to compare such data as in the situation with the cat and the chickadee. With some hints from the teacher (institutionalization) a formulation, verbal or using letters, of some rule of dividing fractions in the sense of finding the ratio between two fractional quantities could be written. The reason why not just ratios are considered in this sequence of problems but comparison of ratios is that division is necessary only when a comparison has to be made. Otherwise one would be satisfied with saying that, for example, for the chickadee, the food weight / body weight ratio is as 9/10 to 15/16, and no further processing of the data (representing the ratio in the form of one number) would be necessary.

PART II - THREE DIDACTIC PHENOMENA: THE 'TOPAZE EFFECT', THE 'JOURDAIN EFFECT', AND THE 'DIENES EFFECT'.

1. The 'Topaze effect'

The name of the 'Topaze effect', which has been described above as 'giving away the answer in the question' in teaching, comes from a play by Marcel Pagnol, written in 1928, in Paris. Pagnol, born in 1895 in Aubagne, near Marseilles, had been a teacher and taught English in various lycées in Southern France and then in Paris. He abandoned this profession in 1922 and devoted himself entirely to writing for the theater and then for the cinema. The play 'Topaze' is set in a private boarding school. Topaze is a teacher in that school. The first scene of the play shows Topaze giving a dictation to a pupil during the recess. The boy, as described by Topaze in Scene III, 'is a conscientious worker but he had some trouble keeping up with the class because no one seems to have taken an interest in him until now', and Topaze decided to help him a little in his free time.

Here is how the first scene starts:

As the curtain rises, Mr. Topaze is giving a dictation to a pupil.… The pupil is a 12-year-old boy. He is turning his back to the public. One can see his ears that stick out and his thin bird-like neck. Topaze is dictating and from time to time bends over the shoulder of the little boy to see what he is writing.

TOPAZE: (*He dictates while he walks up and down*). "Some… lambs… Some lambs… were safe… in a park; in a park. (*He bends over the shoulder of the pupil and continues*) Some lambs… lamb<u>z</u> (*The pupil looks at him, bewildered*). Now, child, make an effort. I am saying lamb<u>z</u>. Were (*he repeats very distinctly*) we-re. That shows that there was not only one lamb. There were several lamb<u>z</u>". (The pupil looks at him, dazed).

(Marcel Pagnol, *Topaze*, Translation and Introduction by Renée Waldinger. Great Neck, N.Y.: Barron's Educational Series, Inc., 1958, p. 2).

Topaze wants the student to succeed; after all, part of the didactic contract is the obligation, for the teacher, to do all he or she can to help the student succeed. But the way he is going about it, is not leading to the student's learning, but to the student's producing a correct answer in spite of not having learned anything.

I found myself caught into the trap of this phenomenon in my linear algebra class. After having introduced the notion of elementary matrices and discussed and illustrated the fact that an elementary row operation on a matrix A has the same effect as multiplication of A, on the left, by an elementary matrix, I gave the students a problem to solve. In the first question of the problem, a 2x2 matrix was given and the task was to find two elementary matrices E2 and E1 such that $E_2\ E_1\ A = U$, where U is the 2x2 identity matrix. The problem, I said, is a typical final examination problem; this immediately arose the students' interest. The students were working on the problem in their exercise books, and I was walking around, looking above the shoulders of the students. A few of them were row reducing the matrix, but many were slowly just copying the matrix, and obviously did not know what to do. After 7 or 8 minutes some 6 students out of the 32 in the class had solved the problem. I let the other students work for another 2 minutes, and then got to the board, and said: 'You have to row reduce the matrix and keep track of the elementary operations; then you just translate the operations into the elementary matrices'. So I told the students how to solve the problem, but did nothing to help them make the conceptual link between elementary matrices and row reduction, or, in other words, to help them understand the theorem. I thought I had explained the theorem sufficiently before, and I simply ignored the fact that, most of the time, explanation does not automatically cause understanding.

2. The 'Jourdain effect'

The name of this phenomenon of teaching alludes to the play by Molière, 'The Cit turned Gentleman' (*Le bourgeois gentilhomme*), which was first acted at Chambord in October 1670, during the reign of Louis XIV. The play is about the sin of vanity in men who endeavor to appear above of what they actually are. In the play, Mr. Jourdain, a good but simple and not too highly educated citizen with no relations to aristocracy, aspires to becoming part of 'la noblesse', if not by blood then at least by manners and education. So he hires a music master, a dancing master, a fencing master, and a philosophy master. The name of 'Jourdain effect' refers to Jourdain's lesson with the philosophy master (Act I, Scene 6), where Mr. Jourdain learns that he had been speaking prose all his life without knowing it:

Mr. Jourdain: … I must commit a secret to you. I'm in love with a person of great quality, and I should be glad you would help me to write something to her in a short billet-doux, which I'll drop at her feet. …

Philosophy-Master: Is it verse that you would write to her?

Mr. Jourdain: No, no, none of your verse.

Philosophy-Master: You would only have prose?

Mr. Jourdain: No, I would neither have verse or prose.

Philosophy-Master: It must one or t'other.

Mr. Jourdain: Why so?

Philosophy-Master: Because, sir, there's nothing to express one's self by, but prose or verse.

Mr. Jourdain: Is there nothing then but prose, or verse?

Philosophy-Master: No, sir, whatever is not prose, is verse, and whatever is not verse, is prose.

Mr. Jourdain: And when one talks, what may that be then?

Philosophy-Master: Prose.

Mr. Jourdain: How? When I say, Nicola, bring me my slippers, and give me my nightcap, is that prose?

Philosophy-Master: Yes, sir.

Mr. Jourdain: On my conscience, I have spoken prose above these forty years, without knowing anything of the matter; and I have all the obligations in the world for informing me about this.

We have to do with the 'Jourdain effect' each time we describe the productions of our students in mathematical terms, which presuppose an elaborate conceptual activity, while having no evidence of such an activity. We say that a student has divided a fraction by another fraction, while it would have been more appropriate to say that the student has changed the sign of division into a sign of multiplication, put the second fraction upside down and used his memory of the multiplication table to produce two numbers separated by a horizontal little line.

We also say that 'a student has solved an equation' to refer to an activity of transforming an expression containing numbers and letters into another such expression, according to certain rules. Most students have no notion of equation, just as Mr. Jourdain had no notion of 'prose'. In a recent research made by an M.T.M. student, Sonia Manago[2], a pair of 16 years-old students were given a set of algebraic expressions to classify according to criteria of their own choice, and their classification proves quite clearly that 'solving an equation' does not mean, for these students, finding those values of the variables for which the equality condition expressed by the equation is satisfied. The girls came up with the following classification, and names for the categories:

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[2] Manago, S. (1999): Two Students' Thinking Process on the Notions of Equation and Solution. *Research Intership Project*, presented in the Master in the Teaching of Mathematics programme, Concordia University, Montreal.

Equal equations: $2=2$, $x+1=x+1$, $3=7-4$, $x^2=9$, $x/5 = 0.2\ x$

Not exactly equal equations: $0.99 = 1$, $8=9$, $3/4 = 5/6$

Formulas: $A = bh/2$, $P = xy/2$, $h = 2A/b$, $y= 2p/x$, $Ax+B = Cx+D$

Equations that equal to zero: $2X = 0$, $2a - 5 = 0$, $2x - 5 = 0$, $X = 0$, $X + 5 = X$

Simple equations: $y = 2x$, $y = ax+c$, $y = x+1$, $y= x^2 + 1$, $x^2 + 1$

Complicated equation: $2x - x$

Y or X equals a fraction: $y = 5/2h$, $x = 5/2$, $y = 5/2\ x$

In the interview with Sonia, one of the students expressed her doubts with accepting x=0 as an equation; she said, 'I don't really think it's anything. I think it's an error!'. Asked, 'what is a solution to an equation', one of the students said, 'The answer to an equation', and the other - 'It's the answer to the equation you are trying to figure out. So if you have a long equation, you do all the steps to get to the answer… It's the final answer'. For the latter student, the solution to x+5 = x was '5=0', because this was 'the final answer'.

2.1 An extreme case of the Jourdain effect: the mathematical program of Zoltan Dienes.

Extreme cases of the 'Jourdain effect' could be observed during the wave of the so-called New Math reforms in 1960s and 70s. Children were given all sorts of 'manipulatives' to play with, toys, dolls, and blocks. When children were sorting toys, their activities were called using the terms of set theory such as 'finding the intersection of two sets'. The case of claiming that some kindergarten or first grade children manipulating cups of yogurt 'constructed the group of Klein' became legendary; it was then used as a joke to ridicule the New Math approaches and unjustified ambitions (p. 139).

A side explanation: the group of Klein

Here is an explanation of what the group of Klein is and what it may have to do with the manipulation of cups of yogurt, for those of you who never heard of it before.

A 'group' in mathematics refers to an algebraic structure of a special kind. An algebraic structure is a set closed under some operations defined on the elements of that set, and satisfying some conditions ('axioms'). From this point of view, the set of all real numbers with the operations of addition and multiplication, is an algebraic structure; because of the properties of these operations (commutativity, associativity, distributivity, existence of a neutral element for each operation, namely 0 and 1, existence of the additive inverses and multiplicative inverses) this structure is called a 'field'. Now, a 'group' is any non-empty set, say G, closed under one operation, say *, which satisfies the following conditions:

(1) the operation is associative

(2) there exists, in G, a neutral element for that operation (i.e. an element e such that for element x in G e*x = x*e = x)

(2) for every element x in G there exists an element y in G such that x*y = e (i.e. every element in G has an inverse in G).

'Group' is, obviously, a theoretical model of such familiar structures as the set of all integers with respect to addition, the set of all real numbers except zero with the operation of multiplication, the set of all real numbers with respect to addition, the set of all rotations of the plane around the same point, the set of all dilations of the plane with respect to the same point, etc.

All these examples are examples of infinite groups, i.e. groups composed of an infinite number of elements. But there exist groups with a finite number of elements, and these were considered appropriate to be introduced to very young children at the time of the New Math reforms. The reformers were supporting their projects by reference to Jean Piaget's psychological theories, and, in particular, to his thesis that the most primitive cognitive 'operational structures' in the child resemble the three most general mathematical structures, which he called 'mother structures', because all other can be derived from them: the algebraic structures, of which the structure of group of transformations is the prototype; the order structures for which the structure of lattice of relations is the prototype; the topological structure of objects invariant under continuous transformations (Piaget, 1969, p. 67[3]). According to Piaget, child's mental development follows the logical development of the theory, from the most general ideas to the most particular ideas, rather than the historical order of the development of mathematics, from knowledge of individual cases, through abstraction, to the general cases. The child is cognitively organizing her experience in transforming objects according to the possibility of reversing and combining these transformations in her mind if not in reality. The child's world in organized in hierarchies of relations between things, events and people that can be modeled by the structure of lattice. The child's geometry is closer to topology than the Euclidean metric geometry which is taught at school; in particular, any closed smooth curve is put in one class by children and called 'a round' - topologically, all such curves are indeed, continuous one-to-one images of a circle.

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[3] Piaget, J. (1969): Psychologie et pédagogie. Éditions Denoël.

From this ‘logical parallelism’ he concluded that teaching of mathematics should be focused, in early grades, on set theory and isomorphisms of structures. The New Math curricular movement fully endorsed this point of view. However, Piaget was warning the reformers against assuming that the child’s cognitive structures are some kind of conscious knowledge, and was even evoking Molière’s Mr. Jourdain to make his point. For him, the discovery of the ‘logical parallelism’ does not solve the pedagogical or didactic problem of teaching mathematics; the problem remains in the form of finding a way of helping the students pass from an unconscious use of the structures to a reflection and theorization of these structures in a symbolic language.

… il se trouve que ces trois structures mères correspondent d’assez près aux structures opératoires fondamentales de la pensée. Dès les «opérations concrètes»… on trouve des structures algébriques dans les «groupements» logiques de classes, des structures d’ordre dans les «groupements» de relations et des structures topologiques dans la géométrie spontanée de l’enfant (qui est topologique bien avant d’atteindre les formes projectives ou la métrique euclidienne, conformément à l’ordre théorique et contrairement à l’ordre historique de la constitution des notions)… S’inspirant des tendances bourbakistes, la mathématique moderne met donc l’accent sur la théorie des ensembles et sur les isomorphismes structuraux plus que sur les compartimentages traditionnels, et tout un movement s’est dessiné qui vise à introduire de telles notions le plus tôt possible dans l’enseignement. Or, une telle tendance se justifie pleinement, puisque précisement les opérations de réunions ou d’intersections d’ensembles, les mises en correspondance sources des isomorphismes, etc., sont des opérations que l’intelligence construit et utilise spontanément dès 7 ou 8 ans et bien plus encore dès 11-12 ans… Seulement l’intelligence élabore et utilise ces structures sans en prendre conscience sous une forme réfléchie, non pas comme M. Jourdain faisait de la prose sans le savoir, mais plus encore comme n’importe quel adulte non logicien manipule des implications, des disjonctions, etc., sans avoir la moindre indée de la manière dont la logique symbolique ou algébrique parvient à mettre ces opérations en formules abstraites et algébriques. Le problème pédagogique subsiste donc entièrement, malgré le progrès de principe réalisé par le retour aux racines naturelles des structures opératoires, de trouver des méthodes les plus adéquates pour passer de ces structures naturelles mais non réfléchies à la réflexion sur de telles structures et à leur mise en théorie (Piaget, ibid. P. 67-68).

But let us come back to our explanation of the notion of ‘group of Klein’. Felix Klein was a German mathematician of the second half of the 19th century, who contributed to the theory of groups, among others. The group of Klein is a four-element group, whose all elements are inverses of themselves. More precisely, any set {e, a, b, c} with an operation * such that the following ‘multiplication table’ holds,

*	**e**	**a**	**b**	**c**
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

One can construct many different models of this group. One is, for example, the set of all plane symmetries that transform a figure made of two parallel segments of equal length, such as the equal sign "=", into itself. There are four such symmetries: the identity transformation, a vertical flip, a horizontal flip and a central symmetry. It is easily verified that each of these symmetries, when repeated, brings the figure back to the initial position and that the combination of any two non-trivial out of them gives the third.

Another example of a model of this group can be constructed using some transformations of the position of a cup of yogurt, which has some inscription or decoration on its side. Suppose at the beginning the cup is standing upright with its decoration in the front. Four transformations of the cup's position are considered: the cup is left untouched (e); the cup is put upside down so that the decoration goes to the other side (a); the cup is put upside down so that the decoration stays on the same side (b); the cup is given a half turn (c). It is easy to verify that these transformations form a group of Klein. As children were playing with the cups of yogurt, they were performing such transformations and their combinations and were, maybe, noticing, that doing the same transformation twice would put the cup back in the original position, and that combining any two of the transformations a, b and c would give the third one, and from this, the 'structurally minded' observers were concluding that the children 'have constructed' the group of Klein. But what the children were actually doing had nothing to do with the identification of the group structure in their manipulations. They were even more in the dark than Mr. Jourdain, because they would not have been able to identify the part of their activity called 'construction of the group of Klein', as Mr. Jourdain could identify when he was speaking prose and when he was not; and they would not be able to produce, as Mr. Jourdain could, further examples of their activity, now sanctified by a scientific term.

3. The 'Dienes effect'

The Dienes effect has more to do with the work of researchers and innovators in mathematics education than with the classroom practices of teachers, like the previous two phenomena. But it is important for us to understand because we are all, in this class, at least part time researchers and innovators in mathematics education. It is important also for our understanding of the theory of didactic situations, because this theory was born, in part, as a result of Brousseau's understanding of what was wrong with Dienes' theory of mathematical instruction.

We start with presenting, briefly, the person of Zoltan Paul Dienes and his theory.

3.1 Zoltan P. Dienes

During the period of the New Math reforms Zoltan P. Dienes, a Hungarian by origin, but fluent in English, French, German and Italian, was a professor at the University of Sherbrooke, in Québec (Servais & Varga, 1971, p. 39[4]). He became well-known to teachers and parents of elementary school children around the world for his blocks (ibid., p. 38, 108) designed for the teaching of position systems of writing numbers in various bases, as well as blocks for the teaching of logic (the set, available commercially in some countries, was often called 'Dienes blocks'). For mathematics educators he became known for his theory of the 'psychodynamic process' of teaching and learning mathematics. When the New Math reforms started being criticized and withdrawn from schools, Dienes left Sherbrooke, and now lives in Wolfville, Nova Scotia, Canada. He still writes books. His latest is "I'll tell you algebra stories you have never heard before" (2002; Upfront publishing, Leicestershire). [the last three sentences were written in 2003].

3.2 The concerns of Zoltan Dienes in mathematics education; the definition of the 'Dienes effect'

Dienes' main problem with mathematics education was no different than that of Brousseau: the discrepancy between the stress, put in mathematics education, on teaching of techniques and the attention awarded to the teaching of the understanding of mathematical ideas (Dienes, 1960, p. 4[5]). But, unlike Brousseau, he did not seek to find the causes of the phenomenon and ways of shifting the pedagogical scales more towards understanding of the actions of the teacher and her interactions with the students and the mathematical content. He 'rule[d] out bad teaching as a regular contributory cause to the present state of affairs' (ibid., p. 5), and set to devise an instructional theory and exemplary teaching materials that would be, in a sense, 'teacher-proof': if administered according to precise instructions of the teacher's guide, they should work with any teacher.

But they didn't; Brousseau has tried them himself and observed them when used by other teachers and discovered that they only 'work' with a committed teacher who, wanting to prove that they do work, would make special interventions and modifications, in order to give meaning to the students' actions and help the students become aware of this meaning. The materials did

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[4] Servais, W. & Varga, T (eds). (1971): *Teaching School Mathematics*. A UNESCO Source Book. Middlesex, England: Penguin Books.

[5] Dienes, Z.P. (1960): *Building Up Mathematics*. London: Hutchinson International.

not induce understanding in students who would work with a teacher who would only just hand out the worksheets, and encourage the students to continue. This belief in the existence of some kind of infallible 'artificial genesis of mathematical knowledge' that would be independent of the teacher's personal investment in the learning process has been called the 'Dienes effect' (p. 37).

3.3 Dienes' instructional theory

The central concept of both Dienes' theory and Brousseau's theory is 'game'. But for Dienes, a game is a special type of play: it is defined as a rule-bound play (Dienes, 1963, p. 24) [6], and it is not assumed that, in a game, one must have winners and losers, and that there are some strategies for winning the game, as in Brousseau's theory. In the theory of didactic situations (TDS), a game is more of a problem situation; the problem is to find a way or a strategy to win the game. The discovery of the finding of this strategy is done through the construction of some new knowledge or a new understanding of some old knowledge, and it is this knowledge that is the true outcome of the game for the winner: she has learned something new.

In TDS knowledge is the outcome of the game but the game does not resemble this knowledge, and the players are not playing with that knowledge; they are using or expected to use that knowledge to win the game. But 'playing with knowledge' or 'playing the target mathematical knowledge' is exactly what the games are all about in Dienes' theory. In order to teach the students a mathematical notion, Dienes proposed to give students sets of manipulatives so structured as to faithfully represent this notion. The students would thus be given some physical representations of the mathematical notion. They would first be allowed to freely play with one representation (Brousseau, TDS, p. 139). Then the play would become more structured: the students' attention would be directed to some properties or actions that would be specific of the target mathematical notion. The students would then be given several other physical representations of the notion and would be directed to focus on the things that were common in all those different 'games' they were playing (the stage of isomorphic games and abstraction). Next, the students would be encouraged and helped to construct a schema of all these games, and formulate it in words or represent it by spontaneously drawn diagrams (the stage of schematization and formulation). The sequence would stop at that stage in earlier grades. In secondary school, the sequence could continue with a symbolic representation of the schema in some accepted mathematical notation, and the formalization of the properties of the target notion in the form of an axiomatic theory (the stages of symbolization, formalization and

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[6] Dienes, Z.P. (1963): *An Experimental Study of Mathematics-Learning*. London: Hutchinson.

axiomatization) (ibid.). The 'notions' that Dienes had mostly in mind were those closely related to Piaget's 'mother structures' and logic: algebraic structures such as groups of transformations or vector spaces, combinatorial problems of ordering, sets and operations on sets, laws of predicate calculus.

3.4 Examples of Dienes' 'lessons'

Example 1: Vector spaces

During a sequence of activities supposedly leading to the identification, by the children, of the structure of vector space, children are given objects (or pictures of objects) of several kinds: two kinds, three kinds, four kinds. The objects are so chosen as to appear in two 'opposite' states. For example, in one activity, they are given cups, gloves and boxes. The cups can be right side up, or upside down. The gloves can be right side out and inside out. The boxes can be open or closed. The children are asked to count how many items of each kind they have, assuming that the rule of this game of counting is that opposite states cancel each other out, so that if a child has, in her set, an open box and a closed box, she puts the pair aside and does not count it (Dienes, 1963, p. 32-33). For example, if a child has 3 closed boxes, 5 open boxes, 2 gloves right side out and 1 glove inside out, 6 cups right side up and 3 cups upside down, her 'book-keeping record' would show:

Boxes	Gloves	Cups
2o	2r	3u

The children would be then encouraged to put their and a neighbor's collections together and count the items. A bookkeeping record of such a 'merger' of possessions could be:

	Boxes	**Gloves**	**Cups**
Jane	2o	2r	3u
Mary	5c	3r	1d
Together	**3c**	**5r**	**2u**

Children would also be asked question such as, 'What would be the state of your record if you doubled/tripled, etc. your possessions?', 'What would happen if your neighbor had the same number of items of each kind but in the opposite state?', etc.

Playing this way with a variety of collections of objects, the children would be encouraged to speak about the common features of all such situations and notice some common features, which could eventually be schematized in the form of, say, a set of book-keeping records regarding a number of kinds of things, each kind of thing appearing in two possible states; the totals of the records can be made by addition and subtraction of numbers in each rubric.

From this kind of plays to the concept of vector space, claimed, by Dienes, to be the target mathematical notion, there is still a long way to go. First of all, in a vector space, one must be able to multiply vectors by any number, not just a whole number. In plays with discrete items such as boxes or gloves, it does not make sense to multiply even by fractions: what is 1/2 of a glove?! But Dienes was not claiming that the target had to obtained in a short sequence of sessions. The sessions could be scattered along one year, two years or more, and the ultimate stage of axiomatization would not be attained till the end of secondary school.

Example 2: A lesson in logic

The description of this lesson, actually conducted by Dienes in a French-speaking classroom in Sherbrooke, comes from (Servais, Varga, ibid. p. 38-40).

Children are given a set of the so-called 'attribute blocks' (also called Vygotsky's logic blocks, or Dienes' logic blocks). Each of the blocks can have one of the 4 kinds of attributes:
Shape: round, triangular, square, rectangular (but not square)
Color: yellow, blue, red
Size: big, small
Thickness: thick, thin

The children work on various problems in groups. Not all groups work on the same problem. In one of the groups, the teacher suggests that the children pick out of the whole set four blocks. Suppose the children have picked the following four blocks:

one triangular, red, small, thick
one square, yellow, small, thin
one square, blue, big, thin
one square, blue, small, thick

The teacher asks the children to formulate sentences of the form 'If… then… ' about this set of blocks and then decide which are true and which are not.
Children formulate sentences and write them down in two columns, one for true sentences and one for false sentences. Here is a sample noted by the observers of the lesson:

True *False*

If red then triangle	If red then square
If triangle then red	If triangle then blue
If blue then square	If square then blue
If thick then small	If small then thick
If large then blue	If blue then large

Now the teacher changes the assignment. He writes the last true sentence on a slip of paper and asks the children to add a fifth block, which would make the sentence false. A child proposes to add a large yellow and thin round block. Then the children are invited to produce sentences that are true about this new set. One of the sentences is: 'If large then thin'. The teacher again proposes to add a block that would make that sentence false. And so it continues. When the children arrive at ten blocks it is not easy to find a simple sentence that is true. So they start the game over game, now with one of the children proposing a true statement about a few blocks, and another falsifying it by enlarging the set.