1999 MATH 645 CONCORDIA UNIVERSITY

LECTURE NOTES ON THE THEORY OF DIDACTIC SITUATIONS – ANNA SIERPINSKA

LECTURE 3

THE NOTION OF 'DIDACTIC CONTRACT'

PART I - IDENTIFICATION OF A PHENOMENON: THE 'DIDACTIC CONTRACT', AND ITS IMPACT ON THE MEANING OF MATHEMATICAL CONCEPTS TAUGHT AT SCHOOL

We have assumed that the didactic situation can be described as a game between a (person in the role or position of the) teacher and the student-milieu system. Every game has its rules and strategies. The rules and strategies of the game between the teacher and the student-milieu system, which are specific of the knowledge taught, are called the 'didactic contract' (p. 41). The assumption that the rules taken into account are pertinent from the point of view of the knowledge taught or aimed at is essential for the definition of the didactic contract. If the rules taken into account have nothing or little to do with this knowledge, but are relevant from the point of view of, say, classroom management, or political correctness, or the general culture, then we might speak of other types of contract, maybe (e.g. pedagogical contract) but not of a *didactic* contract. It is assumed in the Theory of Didactic Situations that 'didactics' refers to the problems of teaching and learning of a *particular* knowledge, not any knowledge in general.

Contrary to games such as chess or bridge, the rules of the didactic contract are not explicit and they can be slightly different from classroom to classroom, culture to culture, and they can even change in the history of a single classroom with the same teacher and the same students. The fact remains, nevertheless, that in every didactic situation there is a didactic contract, and that across different cultures, classrooms and time some rules remain constant, such as, for example, that the teacher is expected to perform teaching actions such as giving the students tasks that are specific to the knowledge he or she aims at, and the student is expected to attend to the tasks given by the teacher.

The rules of the didactic contract are implicit: the teacher and the students do not sign a chart of 'rights and obligations' upon entering into a didactic relation. But they are there and we know that they are there when they are broken. Suppose that one day, in the middle of a class, I suddenly sit down, open a newspaper and start to read. What would you say? That I am not doing my job! I was supposed to teach you something, start speaking to you, assign tasks, etc., and here I am, sitting, and reading a newspaper! You would immediately report on my strange behavior to the chair of the department.

Let us look at another situation: The teacher gives the students the task to write the numbers 38, 24, 49, 46, 51 in an ascending order. The students are left to solve the problem individually, and then the teacher writes on the board the solution to this exercise. Then she turns to the class and asks: 'Why did we put 46 and then 49?' A student answers: 'Because if we only had the other numbers, it would have been too easy'. The teacher answers with some anger in her voice: 'This is not what you are asked for!' ('*Ce n'est pas ce qu'on te demande!*'). Another student says: 'Because 46 is smaller than 49'. Teacher: 'Yes, very good' (and she writes the symbol '<' between the two numbers). We can see that the first student, in her intervention, stepped out of her role of student: she expressed her view on the didactic reasons in designing the task, thus entering the role reserved for the teacher. The teacher felt angry with the 'usurper'.

The historico-epistemological context of the emergence of the notion of didactic contract in the theory of didactic situations

The intellectual background of the idea of 'didactic contract' has been identified and analyzed by Bernard Sarrazy in (Sarrazy, 1995[1]).

'The concept of didactic contract has been introduced by Brousseau in 1978, as a possible cause of the so-called 'elective failure' in mathematics (some students appear to have difficulty only in mathematics at school, while succeeding reasonably well in other subjects). He used it to explain the case of a young boy, Gaël, who was repeating his first grade because of difficulties in mathematics. The observation of the behavior of the boy during the remedial sessions with a tutor showed that, for Gaël, knowing something meant only to be able to repeat certain ritualized actions, modeled by the teacher. He was putting so much effort into finding out what actions the teacher expected him to perform, or into 'uncovering the implicit contract', that he was not able to engage with unraveling the meaning of the mathematical knowledge involved in the tasks posed to him. When asked questions like 'why did you add these two numbers?' he would invariably answer: 'because this is what the teacher said that we have to do', 'this is how I was taught', etc. The boy was not retarded or unintelligent but simply had a notion of the didactic contract that did not allow him to learn any mathematics. The solution of the boy's problems came with changing his notion of the contract by creating didactic situations for him in which the contract was obviously not the same as with his schoolteacher. The boy was no longer told 'Do as I told you!'. Instead the tutor would play a game with Gaël, betting on the possible outcomes of a calculation, and then finding out who was right.

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[1] Sarrazy, B. (1995): Le contrat didactique. *Revue Française de Pédagogie 112*, 85-118.

Describing the problem of Gaël in the sociological and cultural terms of 'didactic contract' rather than in psychological terms of intelligence quotient, or personality, was quite in tune with the spirit of the times. In the second half of the 70s, there was, in the explanations of the school failure, a shift from the macro-sociological theories which would put the blame on the socio-cultural background of the students (e.g. the theories of Basil Bernstein regarding the school failure of working class children), and on the workings of the educational system as a whole (e.g. Bernstein's theory of 'transmission of culture' stressing the conservative character of educational systems), towards micro sociological theories which focused on classroom interactions. This 'interactionist trend' in educational sociology had a well-developed theoretical background in 'symbolic interactionism' and 'ethnomethodology', a research paradigm that emerged in USA in 1930s (the so-called 'Chicago school' of sociology: W.J. Thomas, F. Znaniecki, R. Park, H. Mead, G.H. Blumer). The papers and books of Ernst Goffmann that appeared in mid-70s became very popular in intellectual circles and had a strong impact on the way people started thinking about even their everyday interactions with other people and institutions. Today, the ideas of Goffmann and others working in the interactionist paradigm of sociology have become part of textbook knowledge and any Cegep student enrolled in the Social Sciences program has to learn about them.

It is Goffmann who started identifying the various 'contracts' that bind our interactions with other people in everyday and professional lives. He called them 'frames' (kind of scenarios) that can be played in different 'keys', (e.g. as comedies or tragedies). Applied to the classroom context, one could speak of several identifiable 'frames', such as 'lecturing', 'questioning', 'reprimand', 'praise', etc. The frame of 'school questioning' is very different from the frame of 'asking a question' in a non-didactic situation: in the latter, the person who asks normally does not know the answer! A child who comes to school first time may be quite astonished that the teacher is asking questions she certainly knows the answers for! When she accepts this strange situation as normal, she has already understood the frame and became aware of the existence of a definite didactic contract that binds her own and teacher's behavior.

Paradoxes of the didactic contract

The student expects the teacher to teach, and for some students, this simply means to tell the student how to solve the assigned problems and what answers to give. But if the teacher complies with this expectation, the student will not learn anything, because she will *not* have had to make a *choice* of one strategy among other possible strategies, and of one interpretation among other possible interpretations (p. 41). This is analogous to the information value of a message: if the probability of a message is 1, then the amount of information it carries is 0. Solving a problem one knows how to solve adds nothing to our knowledge.

> So the didactic contract puts the teacher in front of a paradox: everything that the teacher undertakes to make the student produce expected behavior tends to deprive the student of the necessary conditions for understanding and learning of the notion she aims at; if the teacher tells the student what she wants, she can no longer obtain it' (p. 41).

So how can a teacher convey new knowledge to the students? How can a student learn new knowledge?

The Theory of Didactic Situations assumes that learning in a school situation is an adaptation to a milieu. The task of the teacher is then to organize the milieu in such a way that the adaptation will result in the student developing the target knowledge. This is, however, easier said than done.

In the second part of the class we shall be analyzing the 'division problems' of last week, from the point of view of their ability to produce, in the students, an adaptation resulting in an understanding of the operation of division.

PART II - EXAMPLES OF THE IMPACT OF THE DIDACTIC CONTRACT ON THE MEANING OF DIVISION IN TWO GRADE 6 TEXTBOOKS

Results of the activity of classification of 'division problems' in Week 2

Each student received a copy of two sets of 'division problems', one taken from a 1936 textbook (label this set 'I'), and the other - from a 1998 textbook (label this set 'II').
Students worked in four groups of three: G1, G2, G3, G4, for about 20 minutes. Their task was to classify the two sets of 'division problems' according to some criteria of their choice. The last 15 minutes of the class were devoted to a presentation, by each group, of the criteria they chose.

A priori, many different criteria could be chosen. For the 1936 textbook, one criterion could be 'the seasons': there were problems for Halloween, problems for Thanksgiving, problems for Christmas, for Valentines, for spring rains, for summer camping, for summer jobs, etc. Another criterion could be 'types of magnitudes': pure numbers, physical magnitudes (measures and ratios of measures), money. Yet another could be 'kind of numbers involved': whole numbers, fractions, decimals, or natural numbers, integers, rational numbers. One could also classify the problems into one-step problems, two-step problems, etc.

In class, students chose different criteria:
Here are these criteria:
Group G1: classification according to an aspect of the operation of division

Division as the inverse operation with respect to multiplication (e.g. I.6)

Long (synthetic) division

… of whole numbers (e.g. I.11)

… division convertible to division by a whole number (e.g. I.15)

… division convertible to division of whole number by a whole number (e.g. I.20)

Number of parts in a whole (e.g. I.1, I.2, I.4)

Repeated subtraction (e.g. I.4)

<u>Groups G2 and G4</u>

Procedural problems

Disguised procedural problems

Conceptual problems

<u>Group G3</u>

Enigma type problems (the problem can be solved backwards) (e.g. II.3,4)

Problems involving judgment (II.8)

Only group G1 used criteria specific to the notion of the operation of division. Other groups used more general criteria that could be applied to problems related to any mathematical notion. These are important criteria but they are not helpful in the study of the meanings of the operation of division aimed by the textbooks.

Study of the meanings of division conveyed by two grade 6 textbooks

If our aim is to study how the meanings of division conveyed by the problems, it may be a good idea to find out

- What mathematical notions and techniques are *sufficient* to solve each problem,

as compared to

- The mathematical notions and techniques *expected to be used* by students by virtue of the implicit '*didactic contract'*, whose rules are conveyed by the context of the problem.

Analysis of the 1936 textbook problems

1. Alice, Ruth and Mary were the Pop-corn Committee for the Pearson School Halloween party. The girls bought 3/4 of a quart of popcorn and divided it equally among themselves to pop. Each girl took what fraction of a quart of corn to pop?

The problem can be solved by dividing one whole number (3) by another whole number (3): 3 quarters of a whole to be shared by 3 girls.

Here, division is an answer to the question: A quantity Q is divided into n parts: how much of the quantity in one part?

The division does not have to be done numerically. It is enough to visualize 3 things and their distribution among three people:

> 2. Tom and Jimmy were to make a box for a game to be played at the Halloween party. They needed 4 boards each 3/4 ft long. The janitor gave the boys a board 3 ft long. How many boards each 3/4 ft long could they have cut from the 3 ft board?

The problem is not a 'division problem'; it is an 'addition-in-disguise problem'. It can be solved by adding fractions, or even whole numbers. Four boards each 3 quarters of a foot add up to 12 quarters of a foot board, which is a 3 feet board. So the janitor gave Tom and Jimmy enough board to make their box: they could cut their 4 boards from the 3 feet board.

> 3. Henry brought 3/4 of a bushel of walnuts to the party. He divided the nuts into 50 equal shares. Each share was what fraction of a bushel?

One could solve the problem by dividing 3/4 by 50 (a fraction by a whole number), but this is not the only way. Actually, one can do with the notion of sharing a quantity Q (= one quarter of a bushel of nuts) among n (=50) people, and continue with the operation of addition. To answer the question in the problem one needs to understand fractions as 'p parts out of a whole made of q parts'. Here is how the reasoning could go:

Henry had 3 quarters of a bushel of walnuts. He divided each one of these quarters into 50 equal parts. The whole bushel would thus be composed of 200 such parts (a bushel is composed of 4 quarters of a bushel, and 50 + 50 + 50 + 50 = 200). He took one such part from each quarter to make a share (1+1+1): so he took 3 out of 200 parts, or 3/200 of a bushel.

> 4. Children had a peanut relay race. Each team ran 7/8 of a block, and each pupil on the team ran 1/8 of a block. How many pupils were on each team?

This problem can be solved by dividing one whole number (7) by another whole number (1).

The problem is a division problem of the type: There are m things and these things are divided into groups of n things: how many groups can be formed? The division can be performed by repeated subtraction: 7 - 1 - 1 - 1 - 1 - 1 - 1 - 1 = 0, so 7 ones in 7. It can also be solved by counting, after having visualized 7/8 as 7 segments of equal length. It is completely irrelevant that eighths of a block are considered.

> 5. Each of the girls on the Refreshment Committee served 1/2 of a pumpkin pie at the party. The pies had been cut so that each piece was 1/8 of a whole pie. Into how many pieces was each 1/2 pie cut?

The textbook probably expected the students to write 1/2 : 1/8 in answer to this problem, but the problem can be solved by multiplying a number by 1/2: there are 8 parts in each pie, so half of the pie has half the number of parts, i.e. 4. Here the fraction 1/2 is seen as an operator. Again, visualizing and counting could be enough to find the answer, without writing any arithmetic operations.

6. 9/10 ˆ 15/16

The text probably expects the students to use the procedure for the division of fractions:
'When the divisor is a fraction, we must rewrite the example, making two changes:
1. Change the division sign to a multiplication sign
2. Invert the divisor (p. 69)'
introduced in the same section of the book.
This procedure uses the algebraic notion of division as multiplication by the inverse.
This notion is in sharp contrast with the arithmetic notions of sharing, partitioning or counting that were sufficient for solving the introductory word problems, in which numbers did not appear as pure numbers but as quantities of something. In fact, these 'introductory problems' did not build the expected scaffolding for the notion of division now officially introduced, and no conceptual link can conceivably be formed on the basis of just the problems such as the above. Any simple arithmetic problem about concrete quantities can be solved with some arithmetic notion of division. The notion of division as multiplication by the inverse is necessary in the problem of creating a unified theory of number systems, and this is certainly not a task accessible to 6 grade children. If the formal rule is given at this point, the students will never be given a chance to make this link, because the work needed to make it is far too costly in terms of conceptual effort in comparison with the mechanical application of the rule.

Let us try to understand what would be involved in solving the present problem without knowing the 'rule'. I claim that treating this problem as a 'real problem' and giving it to the students prior to giving them the rule, in the form of: 'What would it mean to divide 9/10 by 15/16?' could actually create a situation in which the students would have the chance to finally going beyond their whole number conceptions of division and reorganizing it into a new conception, namely the ratio conception of division. This notion is still an arithmetic notion, albeit a quite elaborate one. The link with the previous conceptions would thus be made, and the leap into the abyss of empty formalism would be avoided.

Let us speculate how a student, who was not given the 'rule' before, could solve this problem. Suppose the student understands this question as: "how many 15/16 in 9/10?". By representing these two fractions as parts of a whole (e.g. a strip of 10 squares), the student may realize that 15/16 is greater than 9/10 by a little and it does not fit into 9/10 a whole number of times. He must, therefore, understand the question differently and thus change his notion of division: not 'how many parts worth 15/16 of the whole in a part worth 9/10 of the whole', because 'how many' presupposes a whole number as an answer, but 'what is the ratio between a part worth 9/10 of the whole to the part worth 15/16 of the whole'? He could then see the whole as composed of, say, 160 little parts; 9/10 of the whole is composed of 9*16 = 144 such little parts, and 15/16 of the whole is composed of 150 such little parts. So the ratio of the 9/10 part to the 15/16 part is the same as 144 to 150, which is the same as 24 to 25.

We can see that in this reasoning, the hypothetical student went from viewing division as partitioning or grouping (how many groups of so many elements) to viewing it as the operation of estimating the ratio of two magnitudes.

Let us notice also that the notion of division as multiplication by the inverse is not necessary for the solution of the problem. Indeed, a generalization of the solution given above could lead to the following algorithm of the division of a fraction by a fraction: to divide a/b by c/d , represent the whole as bd little parts. Then a/b is ad of these parts, and c/d is cb of these parts. Then a/b divided by c/d represents the ratio of ad little parts to cb little parts. Symbolically: a/c : c/d = ad : cb. Of course, the ratio can sometimes be simplified, like in fractions.

7. 5/8 ˆ 15/16

The method elaborated in the previous problem allows us to treat this question as a simple application of the method: we can see the whole as composed of 128 little parts, 5/8 as 80 of these parts and 15/16 as 120 parts; the answer is the ratio of 80 to 120, which can be represented by the reduced fraction 2/3.

8. The cookie recipe that Mrs. White planned to use called for 3/8 cup of chocolate. She had only 1/4 cup of chocolate. What fraction of the full recipe could she have made with that amount of chocolate?

The problem can be solved using the notion of division as representing a ratio of quantities and a notion of the equivalence of one quarter and two eighths: one quarter of a cup is the same as 2 eighths of the cup. The amount of chocolate that Mrs. White had was thus to the full amount needed as 2 to 3. So she could only make two thirds of the recipe. This is a kind of proportional reasoning that does not require the writing of any arithmetic operations, and certainly not the operation implicitly expected by the authors of the textbook, namely 1/4 : 3/8 = 1/4 * 8/3 = 1/1 * 2/3 = 2/3.

9. Divide and put your answer in simplest form: 9/10 ˆ 3/5.

Division as ratio can be used here again; 3 fifths is the same as 6 tenths, so 9 tenths is to 3 fifths as 9 tenths to 6 tenths, i.e. as 9 to 6 or as 3 to 2.

This appears to be quite simple, but this is not what the text expects the students to do: the 'simple form' means the 'mixed number' form, i.e. 3/2 is expected to be represented as 1 1/2. But this representation requires to understand division not arithmetically, as ratio, but algebraically, as an operation on numbers, yielding a number, and not some new kind of entity. Seeing ratios as numbers and extending the four arithmetic operations from whole numbers to fractions is epistemologically quite difficult, because, from the former domain to the latter, these operations change their meaning. One can no longer think of addition as 'bringing together, or of division as sharing, or partitioning a quantity. In the history of mathematics, a unified notion of number has been, de facto, an achievement of the nineteenth century.

The solution of the problem, as it stands, without the implicit 'didactic contract' conveyed by the kind of worked out examples preceding this problem, does not require the above sophisticated notion of division as an arithmetic operation on numbers. The notion of ratio of whole numbers is sufficient. If a student solves the problem with the notion of division as ratio, and is punished by the teacher because she did not transform 3/2 into 1 1/2, this is a sign for her that the rules of the game are not what she taught they were. She does not know what is wrong with her thinking and she is unable to yet understand the notion of division of fractions in an algebraic way. Different scenarios are possible from this point on for the student, some ending in frustration and failure, some - in the student suspending her 'situational' understanding of fractions and happily engaging in the formal game on fractional expressions, changing division into multiplication, inverting, simplifying, extracting the whole part, etc.

We can see now that, in the textbook, the 'algebraic notion of division' does not emerge as a 'conceptual necessity' for solving problems, but is enforced by a 'didactic contract' in a situation of institutionalization, as if it were a law and not a mathematical concept.

> 10. On Halloween the Pine Hill School had some Hard Luck races. The route for the races was in three laps. The first lap was from the school to Five Corners: 1/4 mile. The second lap was from Five Corners to Orr's Sawmill: 7/8 mile. The third lap was from Orr's Sawmill to the school: 3/4 mile. Helen said that the second lap of the route was 3 1/2 times as long as the first. Jane said that it was 3 3/8 times as long. Which girl was correct?

The problem does not require division, but multiplication of fractions and comparison of fractions. A possible line of reasoning could be: If Helen is right then the second lap would have to be 3 1/2 times 1/4 mile. This is equal to three quarters of a mile plus a half of one quarter, i.e. six eighths of a mile plus one eighth, i.e. seven eighths of a mile, so she is right. If Jane were also right then the second lap would have to be 3 3/8 times 1/4 mile. Here the multiplication becomes more complicated, so a mental calculation may be risky and it is better to write down the operations:

3/4 mile + 3/8 * 1/4 mile = (3*8)/(4*8) mile + 3/(8*4) mile = (3*9)/(4*8) mile = 27/32 mile. Now, 7/8 mile is the same as 28/32 mile, so Jane's estimation is inaccurate by 1/32 mile.

If a student chose to solve the problem by dividing 7/8 by 1/4 and checking whether it is equal to Helen's proposal or Jane's proposal, then a new conceptualization of division would be required; 'new' with respect to the conceptualizations needed in the previous problems. To solve the problem with division, the student must first understand the mutually inverse character of division and multiplication: if y is a times x then a is y divided by x. In the problem, y and x are magnitudes - distances, in miles; a is a factor, a 'pure number', so to say. Division could be understood as a ratio, but the ratio must be treated as a number, otherwise the expression '*a* times as long as' wouldn't make sense. The 'must' is a constraint of the problem and the chosen strategy, not of a didactic contract, as in the previous problem.

Here is a possible line of reasoning: 1/4 of a mile is the same as 2/8 of a mile, so the second lap is to the first as 7 eighths to 2 eighths. So the ratio is 7/2 or 3 1/2. Therefore the second lap is three and a half times longer than the first, which is what Helen said, and not what Jane said. So Helen is right.

11. Divide: (a) 912 / 76 (b) 35351 / 431

Question (a) could be solved by repeated subtraction and counting how many times 76 fit into 912 (12 times). But applying the same strategy to (b) would be a bit tedious: 35351 = 431*82 + 9. One could, of course, use some informed guessing: there are 80 four hundreds in 320 hundreds, so let's try 431*80: this is 34480. Adding 431 twice to the result yields 35342 and there is 9 left. In this reasoning, numbers are whole numbers and the division is the 'Euclidean division', which identifies the integer number of times a number fits into another number and computes the remainder. The textbook expects the students to use the long division algorithm and represent the result in the 'mixed number' form: 82 9/431.

> 12. Woods family went to the State Fair. Father and Andy drove to the fair in the truck, taking some cattle to be entered for prizes. Mother and Ruth drove the family car. On the way to and from the Fair, Father used a total of 24 gallons of gasoline and 5 quarters of oil for the truck. The gasoline cost 18 cents per gallon, and the oil cost 30 cents per quart. Father drove the truck 107 7/10 miles in going to the fair and 108 3/10 mile in returning. Besides the cost of the gasoline and oil, the expenses for the truck were $1.00 for repairing a tire. To the nearest cent, what was the cost per mile for the truck for the round trip?

This is 'multiple step' problem with several calculations to be made before the answer can be given, but the ultimate operation is division.

(24 gallons of gas * .18 $/gallon + 5 quarters of oil * .30 $/quarter + 1.00 $) : (107 7/10 + 108 3/10) = 3 cents per mile

Division has to be understood as a ratio here, but the ratio is not a pure number, it is a magnitude, because magnitudes of different kinds are compared: *cost per mile*.

This is again something new with respect to the conceptions needed in the previous exercises, where ratios of homogeneous magnitudes were considered only. In the history of mathematics and physics, the notion of ratio of magnitudes of different kinds was very slow to appear. In Euclid's 'Elements', Book V devoted to the 'theory of proportions', starts with several definitions regarding 'magnitudes' and 'ratio of magnitudes'. Definition 3 states: 'A ratio is a sort of relation in respect of size between two magnitudes of the same kind'. For Euclid, area and length, weight and volume were not magnitudes 'of the same kind'. Also areas of circles and areas of squares were not of the same kind, because figures bound by straight lines and figures bound by curves were regarded as totally distinct entities. Thus, one of the theorems states: Circles are to each other as the squares built on their diameters. In today's terminology and way of thinking about numbers we would say: the ratio of the area of a circle to the area of the square built on its diameter is constant and equal to the number Pi. But Euclid refused to measure the area of the circle with the area of a square and thus he could not obtain, in his system, the number Pi. The requirement of comparing magnitudes of the same kind only has been an obstacle to the development of mathematics for many centuries, but when it was finally overcome and mathematicians started to allow themselves to use ratios of non-homogeneous magnitudes and think of ratios as numbers which could equal to each other and not be just 'proportional', they could invent the notions of velocity (distance per time), acceleration (increase or decrease of velocity per time), and the differential calculus.

The point of all these historical remarks is to stress the non-trivial character of the new conceptualization involved in solving the present problem using division.

> 13. Divide 3/4 by 5/9.

The problem can be reduced to finding a ratio of two whole numbers, 27 to 20: 3/4 is the same as 27/36 and 5/9 is the same as 20/36, so 3/4 to 5/9 is the same as 27 to 20.

> 14. Nancy earned her Christmas money making Christmas cards. She bought 2 sheets of cardboard at 5 cents each, a bottle of drawing ink for 25 cents, and some watercolors for 25 cents. (A) How much did all these things cost? (B) The cardboard sheets were 22 inches by 28 inches in size. She cut each sheet into strips 22 inches long and 5 1/2 inches wide. How many of the 5 1/2 inch strips did she cut from the 2 sheets? How many pieces were too narrow for her to use?

Both questions (a) and (b) can be solved by addition.

A. 5 + 5 + 25 + 25 = 60 (cents)

B. Suppose Nancy cuts across the longer side. One strip of 5 1/2 and another of 5 1/2 make 11 inches; with another pair of 5 1/2 inch strips she has already disposed of 22 inches, so she could cut yet another strip, and thus 5 strips come up to 27 and a half inches, and a strip of one half inch wide and 22 inches long is left over from one sheet of cardboard. So, from the two cardboards she could cut 10 5 1/2 inch strips and had two pieces two narrow for her to use.

> 15. Sally and Ruth decided to make some valentines which would be different from those they could buy in the stores. They bought a sheet of red paper 22 inches by 28 inches. Each girl took 1/2 of it. How many hearts could each girl have cut from her share, if each heart used up to 1 square inch of paper?

The only division that has to be done in this problem is 22:2 or 28:2 and the only interesting question that the students may have about this problem is whether it matters how the paper is cut in two: across the longer side of across the shorter side. Otherwise the problem can be solved by multiplication understood as repeated division, or even by drawing and counting:

15. 155.8 / 2
16. 512 / 32
17. .828 / 6

The textbook regards the above three exercises as practice in applying the algorithm of the long division of decimals. But problem # 15 could be done mentally, by representing it as sharing the amount of 155 and 8 tenths of something between two people. Splitting 155 into 2 gives 77 and a half. Splitting 8 tenths in half gives 4 tenths. So altogether each person gets 77+(1/2 + 4/10) = 77 + 9/10 = 77.9

Problems # 16 and 17 differ from # 15 in that the divisor is greater than the dividend, so division has to be seen as a ratio. In # 16 the dividend is still a number greater than 1, in # 17 it is less than 1, but this may not make a big difference.

How small is 5.12 compared to 32? Changing the unit one can say that 5.12 is to 32 as 512 is to 3200, which can be simplified to 128 to 800 (dividing by 4), and then to 16 to 100 (dividing by 8). The ratio 16 to 100 can be written in decimals as .16.

Problem #17 could be interpreted as an exercise in division of a whole number by a whole number, in its conception as sharing or partitioning: changing the units one could think of .828 as representing 828 grams of something (e.g. chocolate), and then partition or share this amount into 6, obtaining 138 grams. This can then be represented as 0.138 of the kilo.

In each case, as we could see, the long division algorithm for decimals could be avoided at almost no cost.

> 18. During 8 hours on Tuesday there was .96 inch of rainfall. This was an average of what decimal fraction of an inch per hour?

Like in the problems above, long division could be avoided. We could think of .96 of an inch as being 96 'centi-inches', and partition these 96 units into 8 hours (by repeated subtraction, or repeated addition or a combination of multiplication and addition). This gives 12 units per hour. The 12 'centi-inches' can be written as .12 of an inch.

19. Mr. Burns and his family drove their car and trailer to Arrow Head camp to spend a few days. They drove 297.5 miles in 8.5 hours in travelling to the camp. How many miles per hour did they average?

In this problem division has to be thought of as a ratio resulting in a new magnitude and not a pure number. Average velocity in miles per hour has to be calculated: 297.5 / 8.5 [mi/h]. Long division algorithm for decimals can be avoided by a process of rough estimation, multiplication and addition. For example, let's first look at how many times the 8 hours could fit into 297 miles. If our guess is 30, then we calculate 8*30 = 240 and we add 30 halves, which is 15. So 8.5 * 30 = 255. This falls short of 297.5 miles by 42.5. Now 8*5 is 40 and 5 times a half is 2 and a half, so 8.5 * 5 = 42.5 which is exactly the missing remainder of the road. Hence 8.5 * 35 = 297.5 and so the average velocity was 35 mi/h.

20. 7581.6 / 7.8

This problem can be reduced to division of whole numbers, sharing or partitioning 75816 into 78. The result is a whole number, 972.

21. Mr. Mills told Ned and Alice that they could sell vegetables during the summer and keep half of the profits. Mr. Mills helped Ned build a stand. To make the boards below the shelf, they sawed up some 14-foot boards. How many boards 3.5 ft long could they have sawed from each 14-foot board?

The problem could be solved by addition: adding 3.5 + 3.5 + ... until something close to 14 is obtained. In this case, adding 3.5 four times gives exactly 14, so the answer is 4.

22. 3 / 1.25

This problem can be reduced to finding the ratio of 3 to 1.25. Making the unit hundred times smaller, we can think of the problem as finding the ratio of 300 to 125. This ratio can be simplified to 12/5. Representing this ratio in mixed number form we get 2 and 2/5. The ratio 2/5 can be represented as 4/10, so the result is 2.4 in decimal notation.

Analysis of the 1998 textbook problems

1. Look at the two series of operations. How do the divisors and results change? Can you find the missing results?

8 : 8 = 1	3/16 : 8 = 3/128
8 : 4 = 2	3/16 : 4 = 3/64
8 : 2 = 4	3/16 : 2 = 3/32
8 : 1 = 8	3/16 : 2 = 3/16
8 : 1/2 = ?	3/16 : 1/2 = ?
8 : 1/4 = ?	3/16 : 1/4 = ?
8 : 1/8 = ?	3/16 : 1/8 = ?

(a) Add two more operations to each column
(b) By what number should 3/16 be multiplied in order to obtain 3/128?
(c) By what number should it be multiplied to obtain 3/64?
(d) What operations could replace each of these divisions? Do you see a rule?
(e) Write a similar series of operations and give the results.

The notion of division that appears to be aimed at by this series of exercises is the formal mathematical one: to divide by a number means to multiply by its inverse.

Question (a) of this problem could be solved without even looking at the left hand sides of the equalities, and the implicit didactic goal of the exercise could be missed. Questions (b) and (c) appear to aim at re-focusing the students' attention on the left hand sides of the equalities. Then, perhaps, looking for a pattern in the first column the student might come to a generalization: when you divide a whole number a by a fraction 1/c (c being a whole number), the result is the whole number a*c. Looking for a pattern in the second column, the student might come up with two 'rules': (1) in dividing a fraction a/b by a whole number c, the result is a/(b*c), (2) in dividing a fraction a/b by a fraction 1/c , the result is a times the ratio of c to b. To answer question (b), the first rule could be reformulated as, 'a/b : c = a/b * 1/c'.

However, the rules obtained by the students might be a lot more specific than the target general notion of division as multiplication by the inverse.

2. Mom said to Jacek: I bought 6 liters of honey. We'll pour it into 1/2 liter jars. Bring the jars from the cellar. (a) How many jars should Jacek bring? (b) How many jars of 1/4 l would he have to bring? And how many jars of 3/4 liter would he have to bring?

The notion of division as multiplication by the inverse is not necessary to solve the problem. One can solve it by drawing a diagram and counting:

(a) The diagram could be made of 6 pairs of squares, each square representing 1/2 l. So 12 jars are needed for 6 liters of honey.

(b) The diagram could be made of 6 groups of 4 squares, each square representing 1/4 l. So 24 such jars are needed for 6 liters of honey.

(c) The diagramcould be made of 24 squares; partioning these into groups of 3 gives 8 (jars).

3. A quotient is equal to the divisor and it is 4 times larger than the dividend. What is the dividend?

To solve this problem it is not necessary to understand division as multiplication by the inverse. Division as ratio, with ratio understood as number, is enough. But it is necessary to understand the transitive property of equality (if A = B and A = C then B = C), and it is quite useful to be able to represent unknowns by letters. This is more an exercise in logical and algebraic thinking than in division of fractions. Here is how the reasoning might go: Let the dividend be a and the divisor b. We are told that a : b = b and a : b = 4a. This implies that b = 4a (by transitivity of equality). Then a : b = a : 4a = 1/4 (a whole to four times a whole is as 1 to four).

But a : b = b, so b = 1/4 (transitivity of equality). But b is 4 times as large as a, so a is a quarter of b; but a quarter of a quarter is 1/16, so a = 1/16.

However, the notion of division as a ratio is not given a chance to develop in the textbook, so the students can be left with the formal notion of division, and if they are not too good at formal algebraic and logical thinking, they have no chance of solving the problem unless they treat it as a riddle to be solved by guessing or experimenting on their calculators.

4. Find a number which is 4 times as large as the quotient of the numbers 3 1/2 and 2 4/5 increased by 1.

Division as ratio is sufficient to solve the problem: 3 and a half can be seen as 7 halves; 2 4/5 is 14 fifths; finding a common measure (one tenth) leads to seeing 7 halves as 35 tenths and 14 fifths as 28 tenths; so the ratio is 35 to 28 which is the same as 5 to 4. Now the unknown number is 4 times 5/4 + 1, so it is equal to 5 + 4, i.e. the number is 9. The problem is mainly an exercise in translating verbal representations of relations between numbers into arithmetic operations. It is also a preparation for algebraic thinking.

5. 2 1/3 + 3/4 : 1/2

Students guided by this textbook will use the rules of arithmetic on fractions, and the algebraic notion of division as multiplication by the inverse. One can, however, expect a lot of mistakes, because the students have no means to verify, by themselves, if their solution is correct, and the meaningless rules are easily forgotten. If the students could create for themselves a model of the expression, then they could perhaps judge of the validity of their result.

6. -12,8 x (-0,2)

Early introduction of operations on rational numbers (and not just decimals) is another symptom of the algebraic character of the notions of operations on numbers favored by the textbook.

7. 3 1/3 : (-5/6) : (-2)

The main problem here for the students is the order of operations, which is based on pure convention, and not on any rational base.

8. Decide which product is less expensive

(a) Margarine sold in cups of 250 g for 1,32 zl vs margarine sold in cups of 500 g for 2,49 zl.

(b) Yogurt sold in cups of 150 g for 0,93 zl vs yogurt sold in cups of 500 g for 2,60 zl.

Problem (a) can be solved by proportional thinking: since 500 is the double of 250, it is enough to find the cost of two cups of the margarine sold in cups of 250 g for 1,32 zl each: that would be 2,64 zl, which is more than 2,49. So the margarine sold in cups of 500g is less expensive.

Also problem (b) can be solved avoiding division, and using only proportional reasoning: 3 cups of the first product is only 450 grams, but it would cost 3*0,93 zl = 2,79 zl, which is already more than the 500 grams of the second product. So it is less expensive to buy yogurt in the larger cups.