LECTURE 2

THEORY OF SITUATIONS AS A MEANS TO OVERCOME THE 'PROCEDURES VS UNDERSTANDING' DILEMMA IN MATHEMATICS TEACHING

In the 'Introduction' to Brousseau's book, the descriptions and explanations of the situations of action, formulation and validation are based on the notions of 'feedback' and 'dialectic', not the metaphor of 'game' which I used last week to introduce the notion of 'didactic situation'. In fact, the metaphor of game appeared in Brousseau's theory only around 1985-6, while the text of the 'introduction' was based on a research done in the early seventies when the theory was still in diapers. It is nevertheless worthwhile trying to understand this older version of the theory because it contains seeds of many later developments, and gives us a better sense of where the theory comes from.

I'll start, therefore, by trying to explain the meaning of the notions of 'feedback' and 'dialectic' in the context of the theory of situations. This discussion will lead me to showing how the theory could be seen as an emergent of the dialectic between the fatalism of those claiming that nothing but procedures can be realistically taught in mathematics classes and the optimism of those who claim that teaching understanding of mathematics is just a matter of good will of teachers and of good teaching materials. I'll conclude by stating that one of the main claims of the theory is that 'mathematical meaning' - this object of all our didactic desires - is not something given or absolute: it has to be researched, studied. The study of the meanings of the mathematical contents of our teaching is an absolutely fundamental element of research in mathematics education or 'didactics of mathematics', which permeates all and every single question in this domain. Questions related to the teaching and learning of mathematics which can be resolved without the study of the mathematical content of this teaching and/or learning are simply not part of 'didactics of mathematics'; they could be legitimate questions in pedagogy, psychology, sociology, anthropology, communication studies, educational technology, etc.

This class will close with an activity in which you will be invited to experience a phase of such 'study of meaning'. The mathematical concept under scrutiny will be the operation of division.

PART I: THE NOTIONS OF 'FEEDBACK' AND 'DIALECTIC' IN THE EARLY VERSIONS OF THE THEORY OF SITUATIONS

1. Feedback

The term 'feedback' comes from cybernetics, which is defined as the study of communication and manipulation of information in service of the control and guidance of systems (biological, physical, chemical, cognitive, etc.).

Suppose we have two systems, A and B, and system A makes an action aimed at system B. This action results in a *feed forward* of information from A to B. System B may re-act to this information by sending a *feedback* of information to system A. On the basis of this information, the system A may change its way of acting on system B. If the system B does not send feedback to A or A does not react to any information that may come from B, then the A-B super system is called an 'open-loop' system) In the opposite case, the system A-B is called a 'closed-loop' system. (An example of a closed-loop system is placing a heater in a closed room with a thermostat). If the system A acts on B in order to attain some goal, and reacts to its action by attempting to minimize the difference between the goal and the output of B, then the feedback is called negative feedback. If the feedback amplifies the difference, the feedback is called positive feedback.

Example 1

When I present a project of a research paper at a conference, I feed forward some information to the potential readership of the finished paper if published. During the discussion period after my talk I receive feedback from the audience. In the course of the discussion, the differences between my intentions and the audience's interpretations are minimized. This makes me revise my paper so as to better match the interests and ways of understanding of the readers of the paper. In this example, the systems A and B were both cognitive systems, and the feedback was negative feedback. If, in the course of the discussion the differences between my intentions and the audience's interpretations are amplified, the feedback was positive, and I produce a paper which is even less understandable than its previous version.

Let us take some examples from the first phase of the class on 'Race to 20'.

Example 2

Let A be the cognitive system of the high school teacher (*Hteacher*). Let B be the system composed of the students and the didactic milieu of the moment. The central element of this milieu is the game 'race to 20'. The system A feeds forward the information about the rules of the game ‘race to 20’ in a verbal form. In the aim of controlling the accuracy of the transmission of meaning, A ‘puts B in motion’, i.e. the Hteacher makes a student play the game. By acting, system B sends feedback to system A. This feedback carries information back to A on whether the rules of the game have been understood in the intended way. System A re-acts by more verbal feed forward in case B does not act in the intended way (negative feedback). Da capo.

Example 3

If we now enter inside the student-milieu system, we may look at the cognitive system of the student as one system, S, and the system of the game played by two players as the other system, G. G can be seen as a set of all possible two-column tables obtained in playing the game, some of them classified as ‘left player wins’ and other by ‘right player wins’. By entering the game as a player, say, a ‘left player’, S feeds forward some information into G, and obtains a feedback in the form of the ‘wins’ or ‘loses’ verdict. S’s goal is to always get a ‘wins’ verdict, and thus aims at controlling the system B so that it always produces such a verdict. The feedback, in this case, is a negative feedback.

Example 4

Let us now use the notions of positive and negative feedback to compare the effects of traditional and constructivist styles of communication between teacher and the student-milieu systems. In the traditional style of communication, the feedback of the teacher (direct corrections, pointing out of errors, hints) is a negative feedback, aiming at minimizing the differences between the expected output of the student-milieu system and the actual output. In the constructivist style of communication, the teacher’s feedback is a positive feedback: in an attempt to understand the student’s way of thinking the teacher will make the student-milieu system focus on the development of a knowledge that may have little to do with the knowledge intended by the teacher: the difference will be amplified.

2. What does the term ‘dialectic’ refer to?

2.1 Meanings of ‘dialectic’ in the history of philosophy

In the history of philosophy, the term ‘dialectic’ has had many meanings. For example, in Plato’s *Republic*, ‘dialectics’ was synonymous with what we call ‘philosophy’ today, i.e. a systematized intellectual reflection on the nature and genesis of being (i.e. what is and what is not and how can we distinguish between one and the other). In the time of Plato, ‘philosophy’ had a much larger meaning of ‘all knowledge’. In the Middle Ages, the term referred to logic, in the context of a classification of the so called Liberal Arts into *trivium* and *quadrivium*. *Trivium* contained three domains of knowledge about language: grammar, rhetoric, and *logic*, called 'dialectic' at that time. All three 'sciences' provided a technical knowledge necessary for conducting debates. Grammar was the basis for constructing correct sentences and statements; rhetoric served the purpose of persuading the potential opponent that a given statement is true; dialectic or logic was meant to guide the opponents in examining their statements for consistency and truth.

In modern times, the term 'dialectic' in philosophy has been associated with *Hegel's dialectic method* of discussing and solving the various 'dualisms' such as the mind/body, freedom/determinism, universal/particular, the state / the individual dualisms. He claimed that these apparent oppositions can be proved as compatible with each other if seen from the perspective of a third, more general concept. In fact, he saw reality in its evolution as a continuous fight between opposing tendencies which are resolved through a more general tendency, of which the basic two can be thought of as particular cases. In formulating a philosophical argument aiming at resolving a duality he would use the pattern of THESIS-ANTITHESIS-SYNTHESIS. In the 'thesis' he would argue in positive terms for one of the points of view. In the 'antithesis' he would argue in favor of the opposite point of view, stressing the contradictions with the first. In the 'synthesis' he would propose a point of view which would bring the former two together as complementary in the frame of a more general conceptual framework. For example, everyone can see the contradictions between morality from an individual's point of view and morality from the society's point of view. What is good and pleasant for an individual is not necessarily good and pleasant for the rest of the society. Doing only what is good for the society and not what we would really like to do limits our free will, to which we think we are entitled if we are not slaves. For Hegel, these contradictions are overcome by the concept of *ethical life*, which refers to modern institutions such as the family, the civil society and the state. These institutions are a realization of our individual free will. As a consequence, abiding by their rules does not constrain our free will. Philosophers have criticized this resolution of the morality dualisms, but this fact does not abolish Hegel's theory of a dialectic character of the evolution of ideas. According to Hegel, there is no stop after a synthesis has been proposed. A synthesis becomes a thesis which can be subject to a critique leading to an antithesis. The resolution of the opposition gives rise to a new synthesis etc.

2.2 The meaning of the term 'dialectic' in the distinctions between the types of didactic situations

In the theory of situations, the term 'dialectic' refers to the method used by a cognitive system (teacher, student) to manage the contradictions between its expectations concerning the output from the system it attempts to control (the student-milieu system, the milieu, resp.) and the feedback. Feedback is just communication of information. The process of dialectic turns this information into knowledge: out of the contradiction, something positive is coming out, that explains the contradiction and generates ways of avoiding it in the future.

In the situation of action, a student may expect to win by playing 7 on the basis of her belief that 7 is a lucky number. If she loses, she may resolve the contradiction between her expectations and the outcome of the game by concluding that 7 may be a lucky number but not in this game, and starting to notice the properties of the numbers 1-20, specific to the game. She may notice, for example, that when she played 12 she won, so next time around she'll try to play 12. But, with a smart player, she'll lose, and this will be again a contradiction with her previous strategy or 'theory'. Overcoming the contradiction by means of a new 'theory', and continuing this 'dialogue' with the situation, the student will teach herself a method for playing the game so that she wins or is able to predict the outcome before the end. However, in the situation of action, there is no need for these 'theories' or 'rules for action' to be verbalized; they may thus remain largely implicit and unconscious for the student.

The situation changes dramatically in the situation of formulation, where the necessity to communicate forces the students to bring the 'theories' or 'rules for action' to the level of consciousness. An explicitation of a strategy by a student may enter in contradiction with the milieu in two ways:

- via feedback with respect to the form: other students may consider the formulation as unclear;
- via feedback with respect to the validity: the strategy may prove to be ineffective in a game, or may be rejected by an argument of another students.

The resolution of the contradiction in each case brings about some positive new knowledge about the situations: a better way of expressing one's ideas or an improved strategy.

Concerning the validity of the statements of the students, a situation of formulation does not force them to distinguish between validity of a statement and the efficacy of a strategy, nor between a convincing, or authoritative, or forceful statement and its truth value. The minds of the disputants are geared towards action and effective action *in* the situation of the game actually played or to be played in the future; all arguments are subordinate to this goal.

The situation of validation changes the milieu with which the students play in that respect. The objects manipulated now are no longer moves in the game, but *statements about the moves* in the game. A student's 'theory' may fall into a contradiction with another student's 'theory'. In the situation of validation the students will work on deciding which theory is 'true', but the outcome of this work may be a third theory, a clearer, more precise statement. If, in a didactic situation, the object of the students' attention is the validity of certain mathematical statements, then it is considered a 'situation of validation', even if the arguments used to prove or disprove the validity of these statement are not, properly speaking, mathematical proofs. They can be mathematical arguments like 'If I play 17, I win because my opponent can only play 18 or 19 according to the rules of the game, and in each case I can then play 20'. But they can also be empirical arguments like 'If I say 15 I lose because each time I played 15 I lost', or 'Each time I play 14 I win; proof, let's do it!'.

In the situation of institutionalization, the students have to overcome the contradictions or just differences between their own ways of playing the game, speaking about it and justifying their strategies and the teacher's ways of doing those things. In the ideal case, a student is able to resolve the contradictions and bring his or her own knowledge to a higher level of generality.

2.3 Theory of situations as the result of applying the 'dialectic method' to resolve the duality between 'procedural teaching' and 'teaching for understanding' in mathematics education.

Theory of situations was first created as a synthesis aimed at overcoming the opposition, in the traditional teaching of mathematics, between procedural and explicit verbal knowledge (PK) on the one hand, and meaning and understanding (MK), on the other (pp. 128-131[1]). In the traditional teaching of mathematics only the former was the object of the teacher's didactic concern and action. The latter was left to happen by itself, as a function of students' intelligence and practice in the application of the procedures, definitions and theorems in solving exercises and problems. The two types of knowledge were opposed by several features:
- PK is taught, MK is not;
- yet MK is necessary for the acquisition of PK; students who do not understand fail at the examinations;

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[1] see also: Brousseau, G. (1988): Représentation et didactique du sens de la division. In: G. Vergnaud, G. Brousseau, M. Hulin (Eds.), *Didactique et Acquisition des Connaissances Scientifiques. Actes du Colloque de Sèvres, mai 1987*. Grenoble: La Pensée Sauvage éditions, pp. 47-64.

- but they are not taught MK, so they are not to be blamed for their failure in PK;
- teaching methods and curricula are to be blamed for students' failure;
- hence, it is necessary to change the curricula and teaching methods: let's reform the system!

This opposition, unresolved, produces ever new reforms. The slogans of 'teaching for understanding' are at the start of almost every reform, but, somehow, inevitably, the institutionalized teaching of mathematics converges towards the traditional teaching of PK and abandoning of the MK to the students' own devices.

A question for the research in mathematics education is therefore: *What are the objective causes of this convergence?*

Is it the laziness of the teachers that is to blame? Or, rather, the fact that MK requires more time for preparing classes, less manageable classroom situations, a lot more reading of more voluminous students' work? The workload of the teacher may increase exponentially with respect to the traditional teaching to the point where it becomes unmanageable. Moreover, statistically, the results on the official final examinations of the students subjected to the teaching of MK on top of PK do not normally show a spectacular improvement with respect to those taught only PK. Actually, a teaching focused on MK, with its openness to all kinds of interpretations and understandings, may leave some students with conceptions contradictory with the more official meanings of terms and lead to errors. Not only that, but some of these conceptions may become so entrenched that they become real mental obstacles to understanding new knowledge. Therefore, teachers lose motivation to put more work: they see it does not pay off proportionally to their efforts.

Is it the teachers' lack of mathematical and didactic knowledge that is to blame? Or rather the lack of such knowledge in the society and culture? This knowledge needs systematic research, invention and experimentation, and its development should not be left to teachers who have other matters to attend to. There exists a body of mathematics education knowledge, but it does not always translate easily or well into practical knowledge for teachers and their classrooms.

Is it the students' lack of interest in mathematics or lack of intelligence that is to blame? Or is it rather that the existing teaching methods and curricula disregard completely the basic laws of human learning?

These are the naïve speculations and common answers to the traditional opposition between procedures and understanding. These answers are rather fatalistic: they lead to accepting the necessity of the predominance of the PK in teaching mathematics.

Theory of situations is an attempt to find a synthesis via a dialectic between the opposing terms instead of resigning to having to reject one of the terms. It uses the dialectic method on two levels

- the development of the didactic theory (research on mathematics teaching and learning) as a synthesis out of the opposition between the teaching methods focused on PK and those focused on MK,

and

- the development of mathematical knowledge in students as a synthesis of oppositions between
- implicit expectations of the results of an action on a milieu and the actual feedback from the milieu,
- formulation of these expectations and the feedback from the objective and the social milieus,
- the expectations and the validity of these expectations viewed as mathematical statements
- the students' own strategies, interpretations, formulations and arguments and the official mathematical algorithms, definitions, terminology, notation and proof methods.

The didactic theory tries to identify and explain the phenomena of teaching and learning of mathematics; in particular those that are responsible for the convergence of the didactic system towards the PK focused teaching. It tries to capture these phenomena using concepts such as 'didactic contract', 'epistemological obstacle', 'didactic obstacle', 'the Dienes effect', 'the Jourdain effect' or 'the Topaze effect', 'the metacognitive shift', 'didactic memory'. Attempts are also made to apply this theory to 'engineer' didactic milieus which would be less likely to degenerate into PK focused teaching. We shall be seeing both the theoretical and the engineering aspects of the theory in this course.

PART II: AN EXERCISE IN THE STUDY OF MEANING OF A MATHEMATICAL CONCEPT: THE OPERATION OF DIVISION IN GRADE 6

In order to teach not only the procedures but also the meaning of mathematical concepts to the students one has to study this meaning. The meaning of mathematical concepts is not something absolute and given once for all. It changes in time, and it changes in function of the contexts in which it is used, and the purposes for which it is used. A concept like, for example, function, may have a different meaning for an algebraist, for a geometer, for an engineer and for a teacher. Thus the meaning of mathematical concepts is not a given in research in mathematics education: it is, rather, a problem.

Thus, studies of the meaning (or meanings, aspects, etc.) of mathematical concepts are an important part of the theory of didactic situations.

Let us engage in such a study, taking, for example, the notion of division.
We can pose the following questions:
(1). What is the meaning of the operation of division for a research mathematician or for a university teacher?
(2). What is the meaning of the operation of division for an elementary school teacher?

1. Meaning of the operation of division for a research mathematician

For a research mathematician, the operation of division appears in the context not of arithmetic but the study of number and algebraic structures. The question, for them, is not *How to divide?,* but *Is it possible to divide in this particular structure*?

Division is not discussed without the operation of multiplication. The operation of multiplication is thought of as any binary operation defined in a set of elements that need not necessarily be numbers. So one must first have a set of elements with an operation defined in it, called 'multiplication', or something else, it does not matter (you can call it 'operation star'). In order to be able to speak about division, one must have the notion of the *identity element*, or an element e such that $a*e = e*a = a$ for any element a in the set. Then one needs the notion of *inverse*: if a is any element, then it has an inverse if there exists a unique element b such that $a*b=e$. The inverse of a, if it exists, is denoted by a^{-1}. Having all these notions we can now define division: to divide a by b means to multiply a by the inverse of b; or $a/b = a*b^{-1}$. This means that division a/b is defined only if b has an inverse.

In the ordinary field of real numbers, the identity is the number 1, and all numbers except for zero have inverses. In the ring of integers only 1 and -1 have inverses. In the algebra of matrices, multiplication is defined in the well known way (i'th row times j'th column), but there is no unique identity matrix for all matrices. In fact one has to take only square matrices of a given dimension in order to be able to speak about the identity matrix. Only very special matrices have inverses (those with non-zero determinants) and then to divide a matrix by another matrix is to multiply it by the inverse of that matrix.

As another non-numerical example, let us take the set of four elements, denoted {0, 1, x, 1+x} and let's define the operation * in it by the following table:

*	0	1	x	1+x
0	0	0	0	0
1	0	1	x	1+x
x	0	x	1+x	1
1+x	0	1+x	1	x

From this table we can see that all elements except for 0 have inverses. The inverse of 1 is 1, the inverse of x is 1+x, the inverse of 1+x is x. Dividing x by 1+x we have to multiply x by the inverse of 1+ x, which is x. But x * x = 1+x so, if we denote division by #, x # (1+x) = 1+x.

2. Meaning of the operation of division for a grade 6 teacher in 1936 and 1998.

With respect to the question of the meaning of division for an elementary teacher, one could start by looking at textbooks, old and modern, classifying the school 'problems on division' into some types, according to some features that would have to be established.

In this class we shall look at samples of division problems from two textbooks, one American textbook from 1936, and one Polish textbook from 1998, both addressed to 6-graders.

The activity will proceed in several phases:

Phase 1: Individual work. Each student received the two samples and classifies each sample according to some criteria chosen by him- or herself.

Phase 2: Small group work. The class is divided into 4 small groups. Each group agrees on a common classification and writes down the classification criteria explicitly on a sheet of paper.

Phase 3: Presentations: Each small group presents their criteria of classification and gives examples of problems from each category. Every next group stresses what is different in their classification with respect to the previous group.

Phase 4: Whole class discussion on (a) the criteria of classification (b) the differences between the classifications obtained for the two textbooks (c) the differences between the meanings of division between the two textbooks (d) the differences between the meanings of division in the 6th grade and the meaning of division in academic mathematics.

The text that follows will be available to the students only after class.

2.1 Information about the textbooks

2.1.1 Categorization of division problems used by

Knight, F.B., Studebaker, J.W., Ruch, G.M. (1936): *Study Arithmetics. Grade Six*. Chicago: Scott, Foresman and Co.

In this textbook the classification of problems on division is done along one main variable: the kind of entities being divided: numbers, such as fractions, mixed numbers, decimals, or magnitudes or measures. Within each category, subcategories are defined: e.g. in dividing fractions, both elements can be fractions, one of the elements can be a whole number, the divisor can be a fraction with numerator equal to 1.

Titles of chapters and sections related to division

Chapter 3 - Dividing with proper fractions

(Some sections are meant to introduce a concept, to explain; other - to practice a concept, yet other - to apply the concept to 'real life' situations; we arrange the tittles of the sections in indents, like this:

[Explanatory section]

 [Practice section]

 [Application section])

Meaning of division by fractions

Knowing what the divisor is

 Fraction divided by fraction

 Using division of fractions

 Mixed numbers in answers

 Halloween races

 Roman numerals

 Divisors with numerator 1

 The state fair

 Whole number divided by fraction

 A Thanksgiving dinner

 Fraction divided by whole number

 Oyster farming

Chapter 4 - Multiplying and dividing with mixed numbers

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Dividing measures

Making Valentines

Chapter 8 - Multiplying and dividing with decimals

...

Decimal divided by a whole number

Using division of decimals

Remainders in division

Elsie and Bill learn about bees

A new idea in division

Changing fractions to decimals

Dividing a decimal by a decimal

Dividing by .1, .01, and .001

Further work in dividing with decimals

A week in a logging town

Whole number divided by a decimal

Problems using decimals

Final work in dividing decimals

Examples of problems:

1. Alice, Ruth and Mary were the Pop-corn Committee for the Pearson School Halloween party. The girls bought 3/4 of a quart of popcorn and divided it equally among themselves to pop. Each girl took what fraction of a quart of corn to pop?

2. Tom and Jimmy were to make a box for a game to be played at the Halloween party. They needed 4 boards each 3/4 ft. long. The janitor gave the boys a board 3 ft. Long. How many boards each 3/4 ft. Long could they have cut from the 3-foot board?

3. Henry brought 3/4 of a bushel of walnuts to the party. He divided the nuts into 50 equal shares. Each share was what fraction of a bushel?

4. The children had a peanut relay race. Each team ran 7/8 of a block, and each pupil on the team ran 1/8 of a block. How many pupils were on each team?

5. Each of the girls on the Refreshment Committee served 1/2 of a pumpkin pie at the party. The pies had been cut so that each piece was 1/8 of a whole pie. Into how many pieces was each 1/2 pie cut?

6. 9/10 ÷ 15/16

7. 5/8 ÷ 15/16

8. The cookie recipe that Mrs. White planned to use called for 3/8 cup of chocolate. She had only 1/4 cup of chocolate. What fraction of the full recipe could she have made with that amount of chocolate?

9. Divide and put your answer in simplest form: 9/10 ÷ 3/5.

10. On Halloween the Pine Hill School had some Hard Luck races. The route for the races was in three laps. The first lap was from the school to Five Corners: 1/4 mile. The second lap was from Five Corners to Orr's Sawmill: 7/8 mile. The third lap was from Orr's Sawmill to the school: 3/4 mile. Helen said that the second lap of the route was 31/2 times as long as the first. Jane said that it was 33/8 times as long. Which girl was correct?

11. Divide: (a) $76\overline{\smash{)}912}$ (b) $431\overline{\smash{)}35351}$

12. Woods family went to the State Fair. Father and Andy drove to the fair in the truck, taking some cattle to be entered for prizes. Mother and Ruth drove the family car. On the way to and from the Fair, Father used a total of 24 gallons of gasoline and 5 quarters of oil for the truck. The gasoline cost 18 cents per gallon, and the oil cost 30 cents per quart. Father drove the truck 107 7/10 miles in going to the fair and 108 3/10 mile in returning. Besides the cost of the gasoline and oil, the expenses for the truck were $1.00 for repairing a tire. To the nearest cent, what was the cost per mile for the truck for the round trip?

13. Divide 3/4 by 5/9.

14. Nancy earned her Christmas money making Christmas cards. She bought 2 sheets of cardboard at 5 cents each, a bottle of drawing ink for 25 cents, and some watercolors for 25 cents. (A) How much did all these things cost? (B) The cardboard sheets were 22 inches by 28 inches in size. She cut each sheet into strips 22 inches long and 5 1/2 inches wide. How many of the 5 1/2 inch strips did she cut from the 2 sheets? How many pieces were too narrow for her to use?

15. Sally and Ruth decided to make some valentines, which would be different from those they could buy in the stores. They bought a sheet of red paper 22 inches by 28 inches. Each girl took 1/2 of it. How many hearts could each girl have cut from her share, if each heart used up to 1 square inch of paper?

15. $2\overline{\smash{)}155.8}$

16. $32\overline{)5.12}$

17. $6\overline{).828}$

18. During 8 hours on Tuesday there was .96 inch of rainfall. This was an average of what decimal fraction of an inch per hour.

19. Mr. Burns and his family drove their car and trailer to Arrow Head camp to spend a few days. They drove 297.5 miles in 8.5 hours in traveling to the camp. How many miles per hour did they average?

20. $7.8\overline{)7581.6}$

21. Mr. Mills told Ned and Alice that they could sell vegetables during the summer and keep half of the profits. Mr. Mills helped Ned build a stand. To make the boards below the shelf, they sawed up some 14-foot boards. How many boards 3.5 ft. Long could they have sawed from each 14-foot board?

22. $1.25\overline{)3}$

2.1.2 Categorization of division problems used by

Zawadowski, W. et al. (1998): *Matematyka 2001. Podrecznik do klasy 6 szkoly podstawowej*. Warszawa: WSiP.

The categorization in this textbook goes still along the same variable: 'type of numbers divided', but it is a lot less detailed, and less explicit for the student. A new type of numbers appears: 'rational numbers'. Decimals are just the positive rational numbers written in decimal notation. Rational numbers are all numbers which can be written in the form p/q where p and q are integers and $q \neq 0$. The information about a section being related to division (or some other curricular topic) appears only in the table of contents as an additional information addressed mainly to the teacher. The curriculum is, in a sense, hidden in the textbook. This may have been done to avoid compartmentalization of knowledge in students into 'rubrics' such as 'multiplication', 'division', etc.

<u>Titles of sections related to division</u>

5. Instead of dividing…	— division of ordinary fractions
…	
12. About three princes who shared their gold	— division of decimal numbers
13. Minus times minus	— multiplication and division of rational numbers

…

15. A difficult choice — operations in rational numbers

…

25. Which way is the best? — operations in rational numbers

…

33. As close as possible! — operations in rational numbers

Examples of problems

1. Look at the two series of operations. How do the divisors and results change? Can you find the missing results?

8 : 8 = 1	3/16 : 8 = 3/128
8 : 4 = 2	3/16 : 4 = 3/64
8 : 2 = 4	3/16 : 2 = 3/32
8 : 1 = 8	3/16 : 2 = 3/16
8 : 1/2 = ?	3/16 : 1/2 = ?
8 : 1/4 = ?	3/16 : 1/4 = ?
8 : 1/8 = ?	3/16 : 1/8 = ?

- Add two further operations to each column.
- By what number should 3/16 be multiplied in order to obtain 3/128?
- By what number should it be multiplied to obtain 3/64?
- What operations could replace each of these divisions? Do you see a rule?
- Write a similar series of operations and give the results.

2. Mom said to Jack: I bought 6 liters of honey. We'll pour it into 1/2 liter jars. Bring the jars from the cellar. (A) How many jars should Jack bring? (B) How many jars of 1/4 l would he have to bring? And how many jars of 3/4 liter?

3. The quotient is equal to the divisor and it is 4 times larger than the dividend. What is the dividend?

4. Find a number which is 4 times larger than the quotient of the numbers 3 1/2 and 2 4/5 increased by 1.

5. 2 1/3 + 3/4 : 1/2

6. -12,8 x (-0,2)

7. 3 1/3 : (-5/6) : (-2)

8. Decide which product is less expensive
(a) Margarine sold in cups of 250 g for 1,32 zl vs margarine sold in cups of 500 g for 2,49 zl.
(b) Yogurt sold in cups of 150 g for 0,93 zl vs yogurt sold in cups of 500 g for 2,60 zl.

2.2 Brousseau's criteria of classification of division problems

In his already mentioned paper on the 'didactics of the meaning of division'[2], Brousseau identified two sets of criteria for the classification of division problems: one related to contextual variables of the problems, and one related to the concepts involved in the solution of the problems.

(1) Contextual variables

Group 1: type of numbers involved in the division
- natural numbers, decimal numbers, rational numbers, real numbers, etc.
- representation of numbers (fractions, decimals)
- size value of the numbers (<1, >1, small numbers, big numbers)
- the mathematical function of the numbers (cardinal numbers, measures, scalars, linear transformations)

Group 2: type of magnitudes
- physical magnitudes
- dimensions
- definition mode: magnitudes defined as products of magnitudes, quotients of magnitudes, etc.

Group 3: type of didactic situation

Group 4: previously taught techniques of calculation
- sharing manipulations
- repeated subtraction
- factoring
- systematic approximations from above and from below

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2 Brousseau (1988), ibid.

reduction to operations in natural numbers

ways of presenting the computations

(2) Conceptions of division

Group 1: Sharing (finding the number of parts, finding the value of a part)

Group 2: Finding the unknown term of a product

Group 3: Fractioning (fractioning of a unit, commensuration, decimal approximation of a fraction)

Group 4: Linear transformation (finding the value corresponding to 1, ratio of measures, a function 'divide by', etc.)

Group 4: Composition of linear transformations