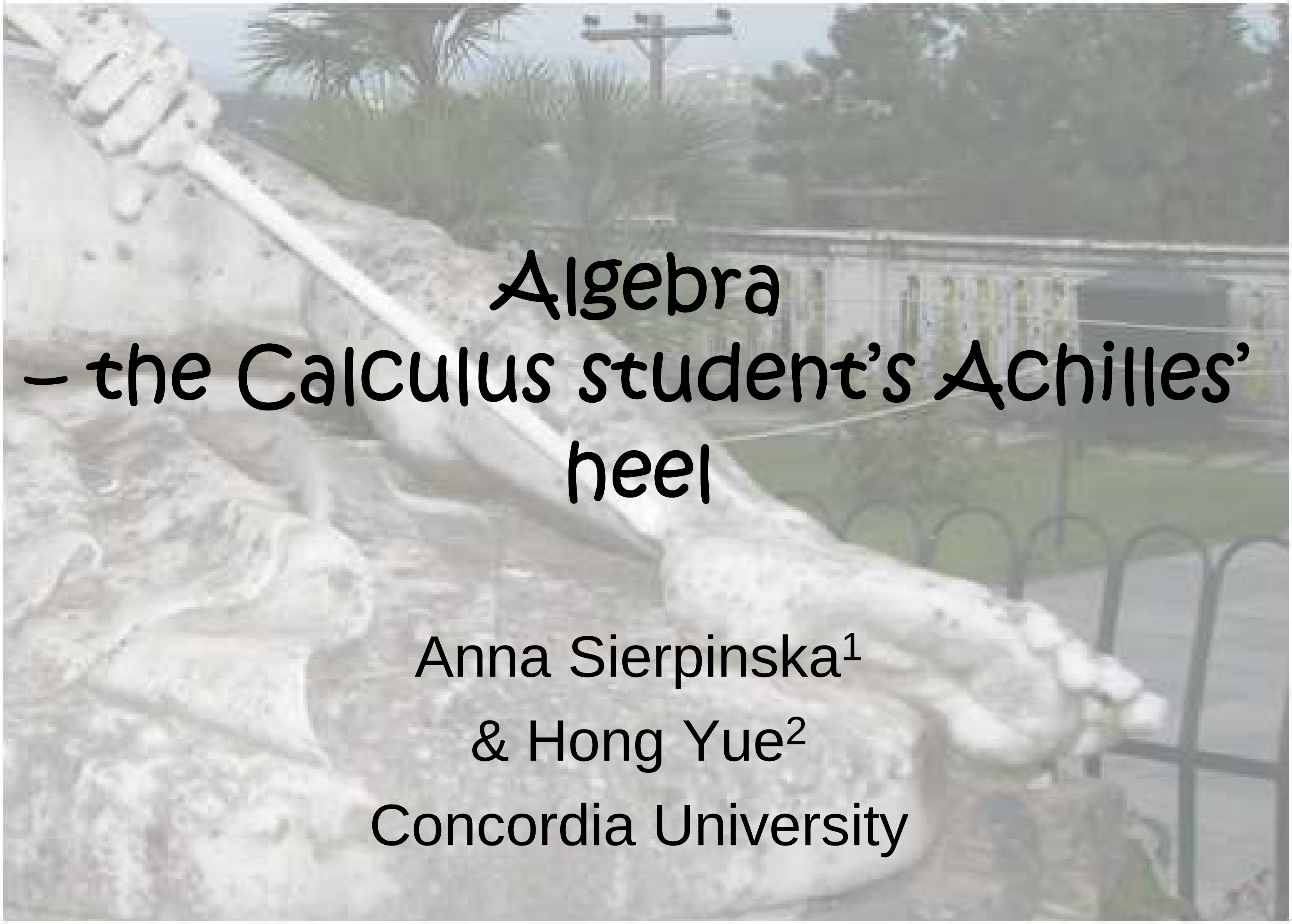
Vanier College – November 11, 2008



Algebra – the Calculus student's Achilles' heel

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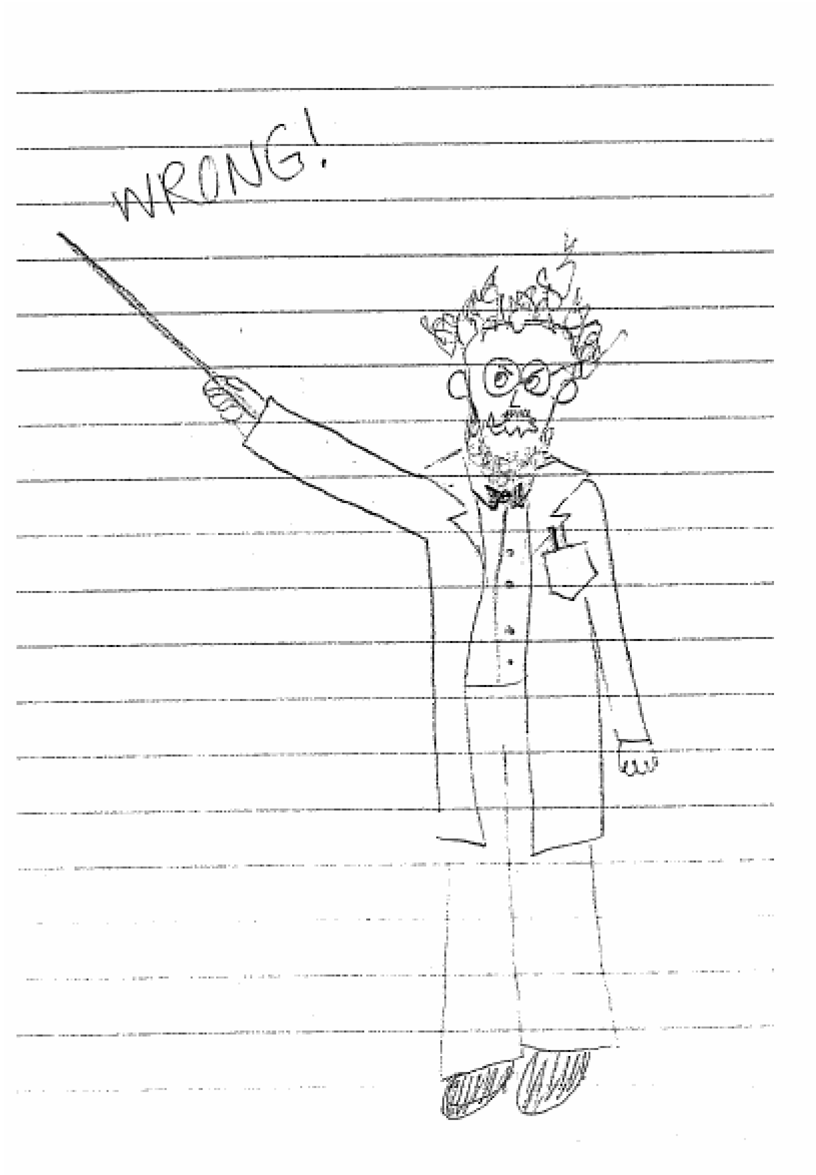


Picture drawn by a student
after a class on inequalities with absolute value

Plan

THEMES:

- 1. Evidence of students' difficulties with algebra
- 2. Theories about sources of difficulties
- 3. Attempts to improve the teaching of algebra

CONTEXTS:

- Algebra, in general – Hong
- The case of absolute value – Anna
- Input from the Audience; general discussion

Theme 1
Evidence of students' difficulties with absolute value

Literature

- Chiarugi, I., Fracassina, G. & Furinghetti, F. (1990). ‘Learning difficulties behind the notion of absolute value’. Proceedings of the Annual Conference of the International Group for the Psychology of Mathematics Education, 1990, Oaxtepec, Mexico, Vol. 3, pp. 231-238.
- Duroux, A. (1983). ‘La valeur absolue. Difficultés majeures pour une notion mineure.’ Petit x 3, 43-67.
- Gagatsis, A. & Thomaidis, I. (1994). ‘Une étude multidimensionnelle du concept de valeur absolue.’ In. M. Artigue et al. (Eds.), Vingt Ans de Didactique de Mathématiques en France (pp. 343-348). Grenoble : La Pensée Sauvage.
- Monaghan, J. & Ozmantar, M.F. (2006). ‘Abstraction and consolidation.’ Educational Studies in Mathematics 62, 233-258.
- Ozmantar, M.F. & Monaghan, J. (2007). ‘A dialectical approach to the formation of mathematical abstractions.’ Mathematics Education Research Journal 19(2), 89-112.
- Ozmantar, M.F. & Monaghan, J. (2007). ‘A dialectical approach to the formation of mathematical abstractions.’ Mathematics Education Research Journal 19(2), 89-112.
- Perrin-Glorian, M.-J. (1995). ‘The absolute value in secondary school. A case study of ”institutionalisation process”’. Proceedings of the 19th meeting of the International Group for the Psychology of Mathematics Education, Vol. 2, pp. 74-81.

Duroux, 1983

Question	Frequency of correct answers (N=46)
What is the maximum of the set E={19, -1, -1253, 27, -3}	92%
If a is a real number, what is the sign of $-a$?	48%
If $\lvert -a \rvert = 1$, what is the value of a?	11%

Description of students' solutions

If $2a + 3 = 1$, what is the value of a?	80%	
If $\lvert a + 2\rvert = 4$, what is the value of a?	22%	1. $\lvert a+2\rvert = a+2$ or $a - 2$ (the abs.val. changes + into –) 2. If $a > 0$ then $\lvert a+2\rvert=a+2$. If $a < 0$ then $\lvert a+2\rvert=-a+2$ (abs.val. applies only to the variable) 3. If $\lvert a + 2\rvert = 4$ then $a = 2$ or $a = -2$ (identifies $\lvert a + 2\rvert$ with $\lvert a\rvert + 2$) 4. If $\lvert a + 2\rvert > 0$ then $\lvert a + 2\rvert = a + 2$. If $\lvert a + 2\rvert < 0$ then $\lvert a + 2\rvert = - a - 2$.
Solve the equation $\lvert x + 1\rvert + \lvert x + 3\rvert = 5$	20%	
Solve the equation $\lvert\lvert 2x +3\rvert - 4\rvert = 6$	0%	

Simplify the fraction $\frac{\lvert a \rvert}{a}$	34%	
Define the function $f: x \rightarrow \lvert x + 1 \rvert$ without using the absolute value	? (70% do not even attempt the question)	
If the integer a is less than the integer b is it possible to have $\lvert a - b \rvert = 2$?	11%	Sample answer: No, because if $a < b$ then $a - b < 0$ while absolute value is always positive.

Chiarugi, Fracassina & Furinghetti, 1990

Is there an *x* such that the following equality is true?

$$\sqrt[4]{x^2} = \sqrt{-x}$$

where $x \in R$

---

14 y.o.: 39% 17 y.o.: 57% 19 y.o: 39%

N=273

Documents from our own research

Preliminary findings have been published in

- Sierpinska, A.: 2007, ‘I need the teacher to tell me if I am right or wrong’, *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education, Seoul, South Korea, July 8-13*, *2007*, Vol. 1, pp. 45-64.

1. Calculate: $||16-24| - |7-56||$

In each of the following exercises,
find all values of x
for which the given inequality is true

2. $|x - 1| < |x + 1|$

3. $|x + 3| < -3|x - 1|$

4. $|2x - 1| < 5$

5. $|2x - 1| > 5$

6. $|50x - 1| < |x + 100|$

Question 4: $|2x - 1| < 5$

$$2x - 1 < 5$$

$$2x < 6$$

$$x < 3$$

$$-2x + 1 < 5$$

$$-2x < -4$$

$$2x < 4$$

$$x < 2$$

$$x < 2$$

Solution of VA-S-3

2) $|x-1| < |x+1|$

$y - 1 \geq 0$ $x + 1 \geq 0$

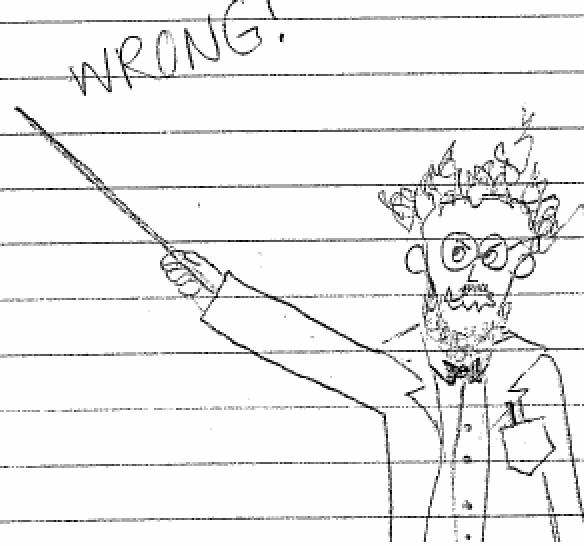
$x \geq 1$ $x \geq -1$

3) $|x+3| < -3|x-1|$

$x + 3 \geq 0$ $-3x - 1 \geq 0$

$x \geq -3$ $\frac{-3x}{-3} \geq \frac{1}{-3}$

$x \leq -\frac{1}{3}$



4) $|2x-1| < 5$

$2x - 1 \geq 0$

$x \geq \frac{1}{2}$

5) $|2x-1| > 5$

$2x - 1 \geq 0$

$x \geq \frac{1}{2}$

exercises

#1 $||16-24|-|7-56||$

$||-8|-|-49||$

$|8-49|$

$|-41|$

$=41$

#2 $|x-1|<|x+1|$

$|x-1| = x-1 < |x+1| = x+1$

$x-1 \geq 0$ $\quad$ $x+1 \geq 0$

$x \geq 1$ $\quad$ $x \geq -1$

#4 $|2x-1|<5$

$2x-1<5$

$2x<6$

$x<3$

?

#5 $|2x-1|>5$

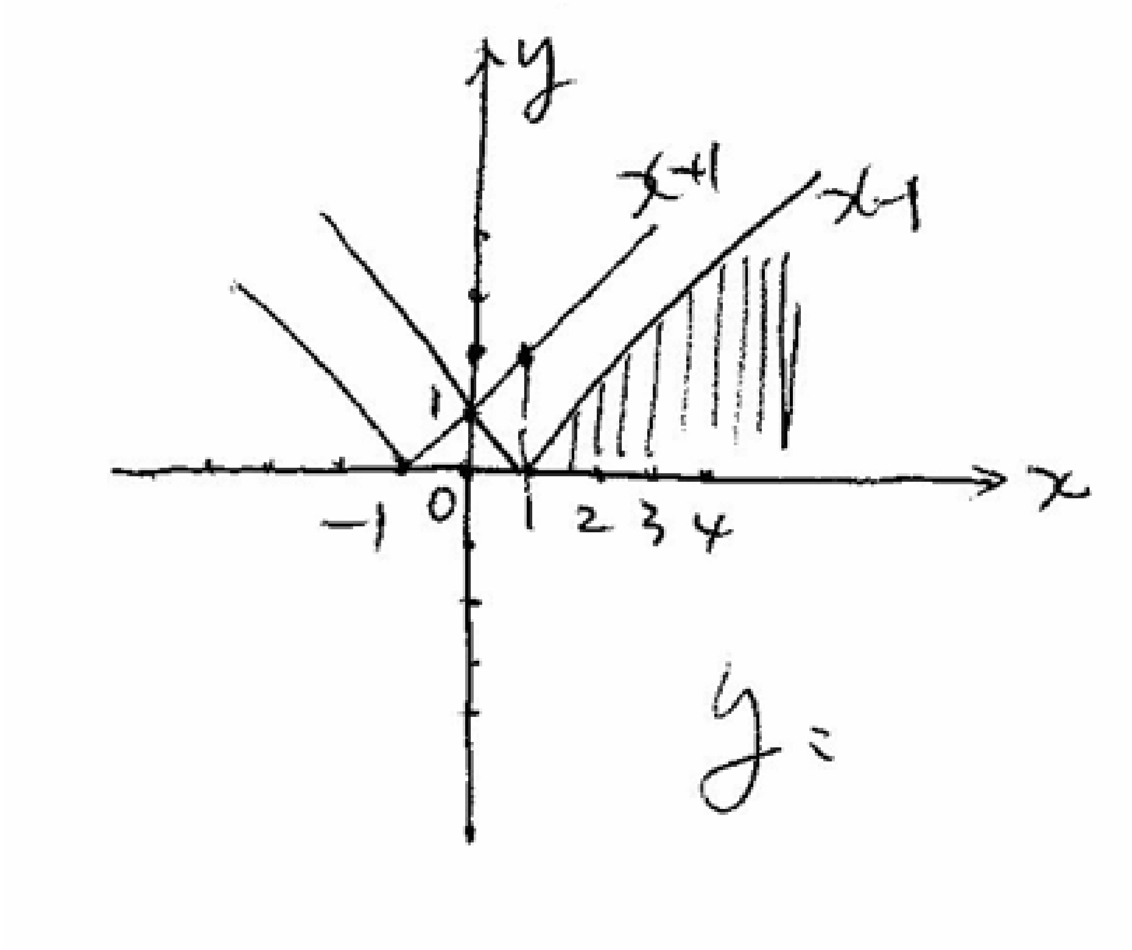
$2x-1>5$

$2x>6$

$x>3$

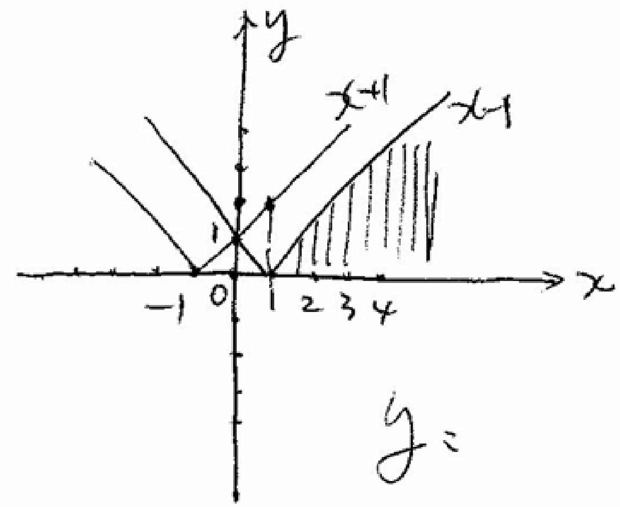
?

$|x - 1| < |x + 1|$



PA-U-2's result in Question 2 is "$x > 1$"

$|x - 1| < |x + 1|$



PA-U-2: So, y is here; it’s always higher

Interviewer: Yes, but it’s also higher from 0 to 1, isn’t it?

PA-U-2: No, you should compare the same direction

Interviewer: Same direction?

PA-U-2: Yes

Interviewer: The same sign of the slope?

PA-U-2: The same sign of the slope

Theme 2.
Sources of students' difficulties with absolute value

Literature

- Cauchy, A.-L. (1821/1968). *Cours d'Analyse de l'Ecole Royale Polytechnique: 1.T., Analyse algébrique* (Reprografischer Nachdruck der Augs. Paris 1821). Darmstadt" Wissenschaftiche Buchgesellschaft.
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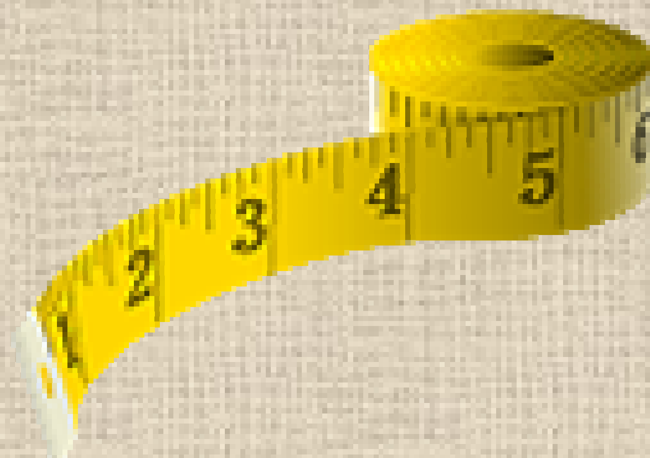
- Skemp, R.K. (1978). 'Relational vs Instrumental Understanding'. Arithmetic Teacher 26(3), 9-15.

Epistemological obstacles to understanding absolute value

- Thinking of number as absolute measure
- Thinking of functions as algebraic expressions
- Thinking of letters as representing fixed but unknown numbers

Thinking of number as absolute measure



Number = absolute measure

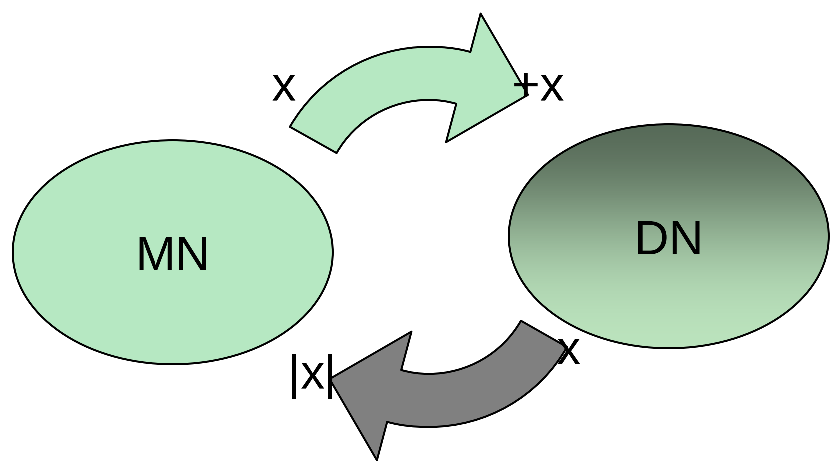
- In a microbiology research lab, there is a problem with the freezer. The freezer's temperature has been set to keep -80 degrees Celsius, but suddenly the temperature changes.
- The first time someone notices it, it is -50; some time later it is -45.
- The microbiologists in the lab lament, "The temperature has *dropped* considerably and keeps *dropping*"!

- A mathematician overhears them, and is confused, "So what are you crying about? Isn't it what you want the temperature to do?"

16:00	-80º
16:30	-50º
16:45	-45º

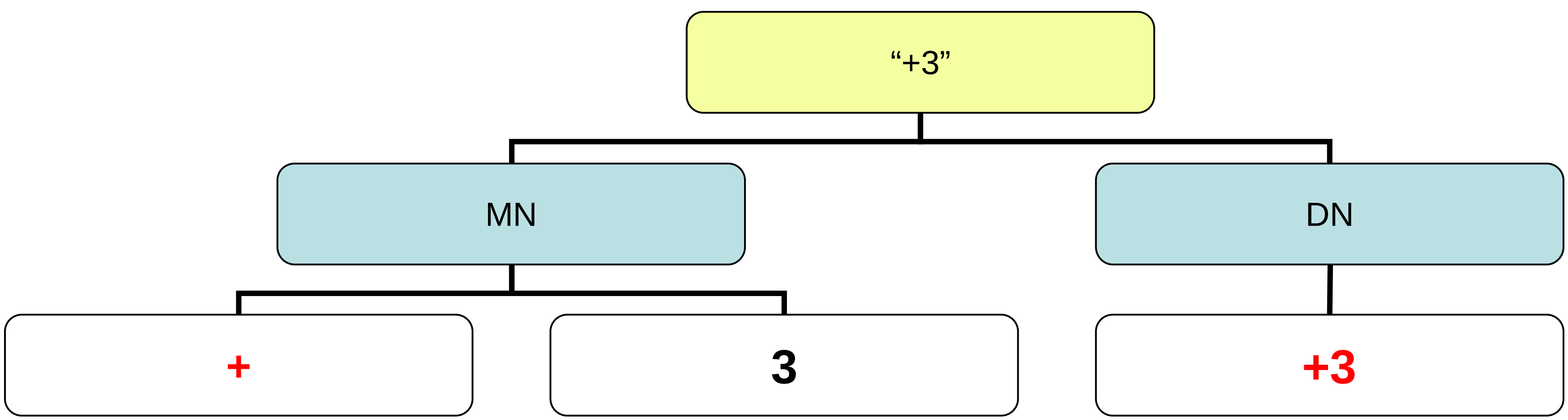
Understanding the notion of absolute value requires the concept of directed number

- Taking into account the direction of change (increase or decrease) results in a new concept of number – let's call it 'directed number' – of which the absolute measure concept is only a particular aspect, namely "the numerical (or absolute) value".
- Seeing the absolute measure numbers as isomorphic with a subset of the "directed numbers" is not immediately obvious or natural for a mind used to thinking of numbers as absolute measures.



Consequences of the MN conception of number

- The two functions account for an isomorphism between MN and a part of DN.
- This isomorphism – as any isomorphism in mathematics – points, at once, to a structural *similarity* between MN and DN, and a *difference* in the nature of these objects.
- From the perspective of DN, a symbol like, for example “+3” represents one single whole, a number in itself. From the perspective of MN, this symbol represents two objects: a number (3) and a sign (+).



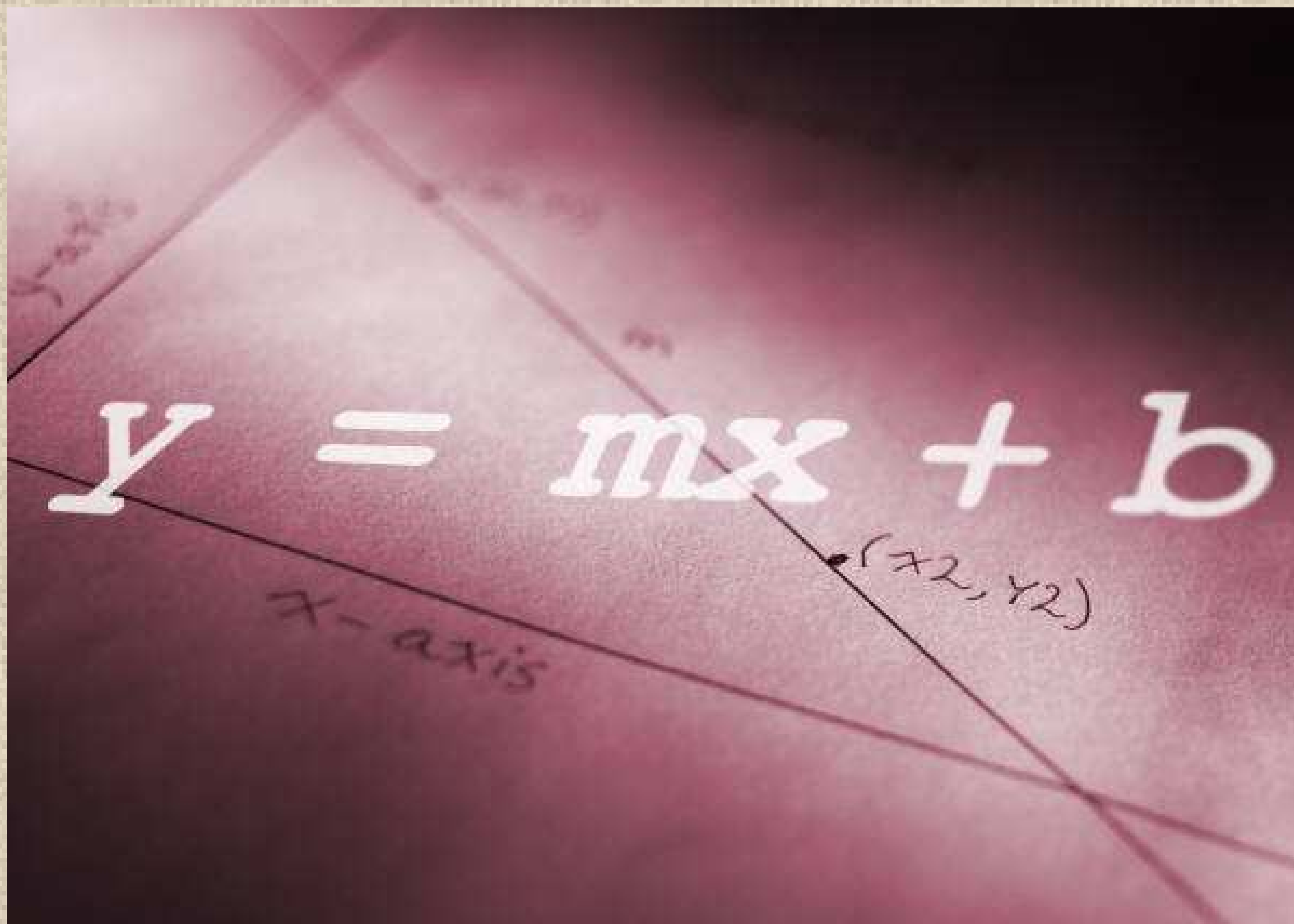
Consequences of the MN conception of number

- From the point of view of MN, “x”, which appears to represent a single entity, must refer to absolute measure, a number without a sign.
- The symbol “$-x$” then necessarily refers to a negative number.

Cauchy, 1821, Cours d'Analyse

- "We will always use the word "number" in the sense in which it is used in Arithmetic, deriving numbers from the absolute measure of magnitudes, and we will use the word "quantity" uniquely to real positive or negative quantities, i.e. to numbers preceded by the signs + or -. Moreover, we will regard quantities as representing increase or decrease, so that, if we are interested only in comparing a given magnitude to another magnitude of the same kind taken as a unit, then we will represent the magnitude by a number, and if we see the magnitude as representing an increase or decrease of a fixed magnitude of the same kind, then we will represent it by the same number preceded by the sign + or the sign -."(our translation)

Thinking of functions as algebraic expressions



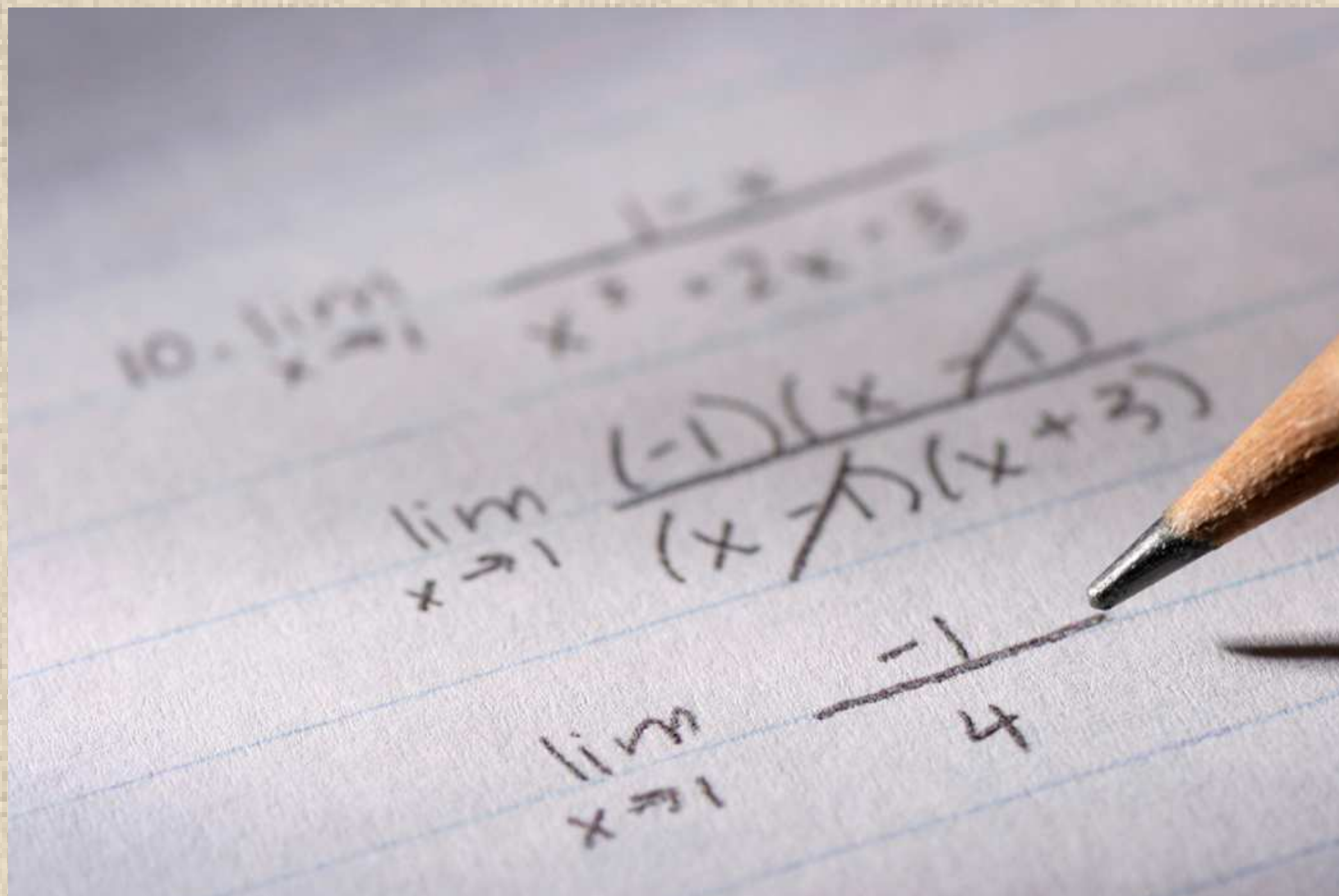
Is this one function or two?

$$|x| = \begin{cases} x & if \quad x \geq 0 \\ -x & if \quad x < 0 \end{cases}$$

Is an alternative definition a solution to the misconception?

$$|x| = \sqrt{x^2}$$

Thinking of letters as representing fixed numbers



“variable” = an arbitrary element of a set

- The notion of letter as representing a variable is necessary to formulate *conditional* statements about values of expressions

Küchemann, D.: (1981). ‘Algebra’. In K.M. Hart (Ed.), Children’s Understanding of Mathematics: 11-16. London: John Murray.

Theme 3.
Teaching approaches to absolute value

Literature

- Ahuja, M. (1976). ‘An approach to absolute value problems.’ Mathematics Teacher 69.7, 594-596.
- Arcidiacono, D.B. & Wood, D. (1981). ‘A visual approach to absolute value.’ Mathematics Teacher 76(3), 197-201.
- Brumfield, Ch. (1980). ‘Teaching the absolute value function.’ Mathematics Teacher 73(1), 24-30.
- Denton, B. (1975). ‘The modulus function’. Mathematics Teaching 73, 48-51.
- Dobbs, D.E. & Peterson, J.C. (1991). ‘The sign-chart method for solving inequalities.’ Mathematics Teacher 84.8, 657-664.
- Eastman, P.M. & Salhab, M. (1978). ‘The interaction of spatial visualization and general reasoning abilities with instructional treatment on absolute value equations’. *Journal for Research in Mathematics Education 9.2*, 152-154.
- Horak, V.M. (1994). ‘Investigating absolute value equations with the graphing calculator.’ Mathematics Teacher 87(1), 9-11.
- Kiser, L. (1990). ‘Interaction of spatial visualization with computer-enhanced and traditional presentations of linear and absolute value inequalities.’ Journal of Computers in Mathematics and Science Teaching 10(1), 85-96.
- Parish, C.R. (1992). ‘Inequalities, absolute value and logical connectives.’ Mathematics Teacher 85.9, 756-757.
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- Priest, D.B. (1981). ‘Inequalities, signum functions and wrinkles in wiggle graphs’. ERIC document ED 209 077; category of Guides for classroom teachers.
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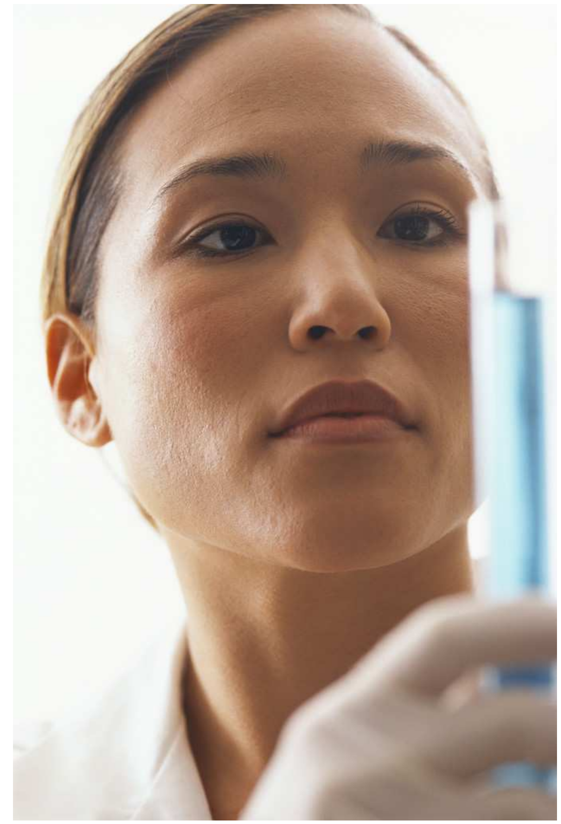
- Stallings-Roberts, V. (1991). ‘An ABSOLUTEly VALUEable manipulative.’ Mathematics Teacher 84.4, 303-307.
- Wei, Sh. (2005). ‘Solving absolute value equations algebraically and geometrically’. Mathematics Teacher 99(1), 72-74.
- Wilhelmi, M.R., Godino, J.D. & Lacasta, E. (2007). ‘Didactic effectiveness of mathematical definitions. The case of absolute value.’ *International Electronic Journal of Mathematics Education 2*(2).

Why teach absolute value at all?

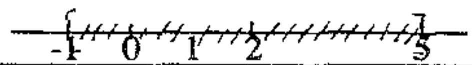
- Isn’t absolute value just a type of secondary school math problems where the absolute value notation is mere shorthand used to make the problem more difficult and requiring the study of cases?
- A lithmus test of mathematical maturity?



Scepticism about the educational value of absolute value

Perrin-Glorian, 1995 (France):

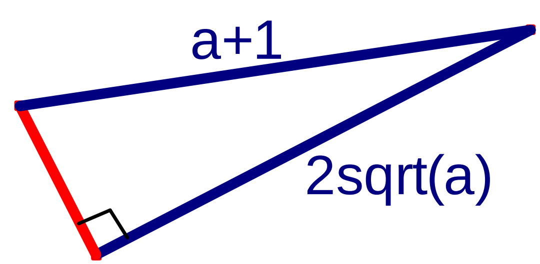
- It is hard to find problems for which absolute value would be a tool, without explicitly mentioning it.
- In fact, the only exercises where students are obliged to use absolute value are problems of translation from one language to another.
- So the difficulties with absolute value are difficulties of mathematical language and logic since we seek logically equivalent expressions when translating from one language to another.

valeur absolue	distance	droite des réels	encadrement	intervalle
$\|x-2\| \leq 3$	$d(x; 2) \leq 3$		$-1 \leq x \leq 5$	$x \in [-1 ; 5]$
$\|x\| \leq 7$				
$\|x-1\| \leq 3/2$				
	$d(x; 5) \leq 9/2$			
			$1 \leq x \leq 4$	
				$x \in [-7 ; -2]$
$\|x+3/2\| \leq 2$				
			$1,4 \leq \sqrt{2} \leq 1,5$	
				$\sqrt{3} \in [1,73 ; 1,74]$
$\|\pi - 22/7\| \leq 10^{-2}$				

Absolute value as a mathematical tool?

- Chiarugi, Fracassina & Furinghetti, 1990

Question 9: In a right-angled triangle one side is $2\sqrt{a}$ cm and the hypotenuse is $a + 1$ cm, with $a > 0$. Find the length of the other side.



MATH 206 - Precalculus

- Absolute value function: Useful in Cal as example of a continuous but not everywhere differentiable function

- Absolute value inequalities prepare students for
  - The epsilon-delta definition of limit
  - Convergence tests

A teaching experiment at Concordia

- TA – “theoretical approach”
- PA – “procedural approach”
- VA – “visual approach”

to teaching inequalities with absolute value.

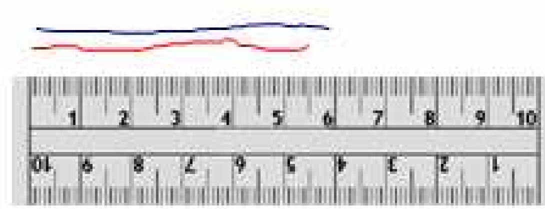
18 Concordia students in total, 6 in each approach

19 Secondary-5 students: 10 in PA, 9 in VA

A LESSON ON ABSOLUTE VALUE

Jane and Joe are measuring
the circumference of a dime with a string.

Jane's result is: 55 mm
Joe's result is: 58 mm



Tom knows the true length of the circumference: 56 mm.
He calculates the difference between the true length
and the measurements:

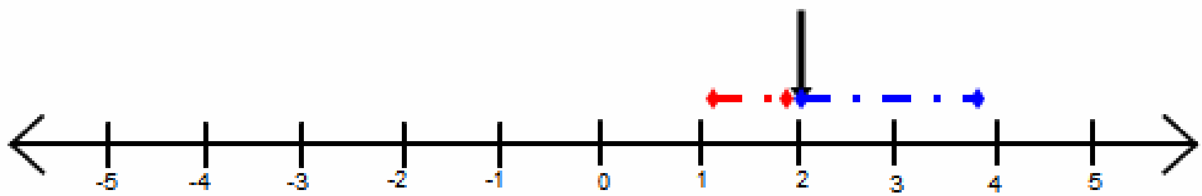
$56 - 55 = 1 \qquad 56 - 58 = -2$

He says: Since $1 > -2$ then Jane made a bigger mistake than Joe.

Do you agree with Tom?

Sometimes we are not interested in knowing whether a measurement was less than or greater than the true value but only in the MAGNITUDE of the difference.

We call this magnitude THE ABSOLUTE VALUE of the number obtained as the difference.



$|2 - 1| < |2 - 4|$

The absolute value of 2 - 1 is equal to 1.

$$|2 - 1| = |1| = 1$$

The absolute value of 2 - 4 is equal to 2.

$$|2 - 4| = |-2| = 2 = \text{the opposite of } -2 = -(-2)$$

The notion of **absolute Value of a number**
is an abstraction from the Context of Comparing
the magnitudes of measurement errors
which involves Calculating the absolute Values of differences between
numbers .

But when we write, for example,

$$|7| \quad \text{or} \quad |-7|$$

we really mean,

$$|0-(-7)| \quad \text{or} \quad |0 - 7|$$

both of which are equal to 7.

We Can say that the absolute Value of 7 is equal to itself,

and the absolute Value of -7 is equal to the opposite of itself: -(-7)

In general, the absolute value of a number x is equal to x for $x > 0$, and to $-x$ for $x < 0$.
As for the number zero, we decide that $|0| = 0$.

Symbolically, we usually write this definition as follows:

$$|x| = \begin{cases} x & \text{if } x \text{ positive or } 0 \\ -x & \text{if } x \text{ is negative} \end{cases}$$

Thus, the absolute value of a number is never negative;
it is positive or zero.

Worked out example

To find all numbers x such that $|x - 1| < |x + 2|$

TA

Let's reason, using the definition.

$$|x-1| = \begin{cases} x-1 & \text{for } x-1 \geqslant 0 \\ -(x-1) & \text{for } x-1 < 0 \end{cases}$$

$$|x+2| = \begin{cases} x+2 & \text{for } x+2 \geqslant 0 \\ -(x+2) & \text{for } x+2 < 0 \end{cases}$$

There are 2 cases for each absolute value in the inequality...

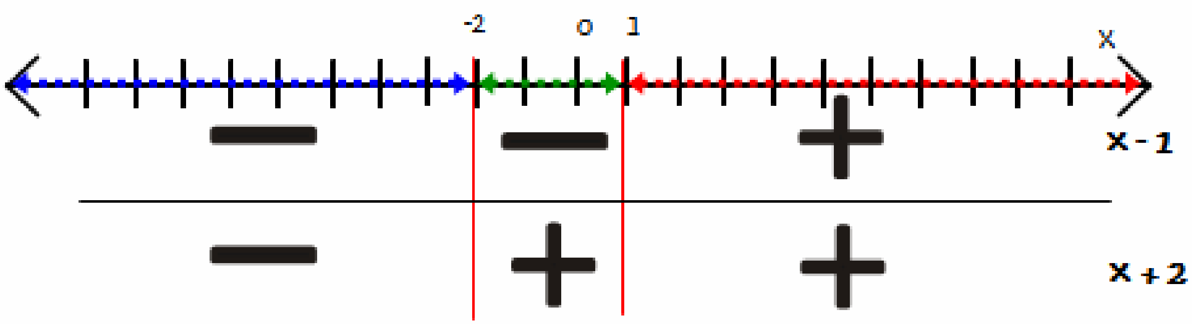
... so we have four cases altogether to consider.

1. $x-1 \geqslant 0$ and $x+2 \geqslant 0$ and $x-1 < x+2$
2. $x-1 \geqslant 0$ and $x+2 < 0$ and $x-1 < -(x+2)$
3. $x-1 < 0$ and $x+2 \geqslant 0$ and $-(x-1) < x+2$
4. $x-1 < 0$ and $x+2 < 0$ and $-(x-1) < -(x+2)$

PA

Since there are infinitely many numbers, it is not possible to find all numbers x for which the inequality $|x - 1| < |x + 2|$ is true just by guessing.
We need a systematic way of finding all these numbers in a finite time.

Here is a method.



On the number line,
mark the number for which $x - 1 = 0$, i.e. the number 1;
mark also the number for which $x + 2 = 0$, i.e. the number -2.

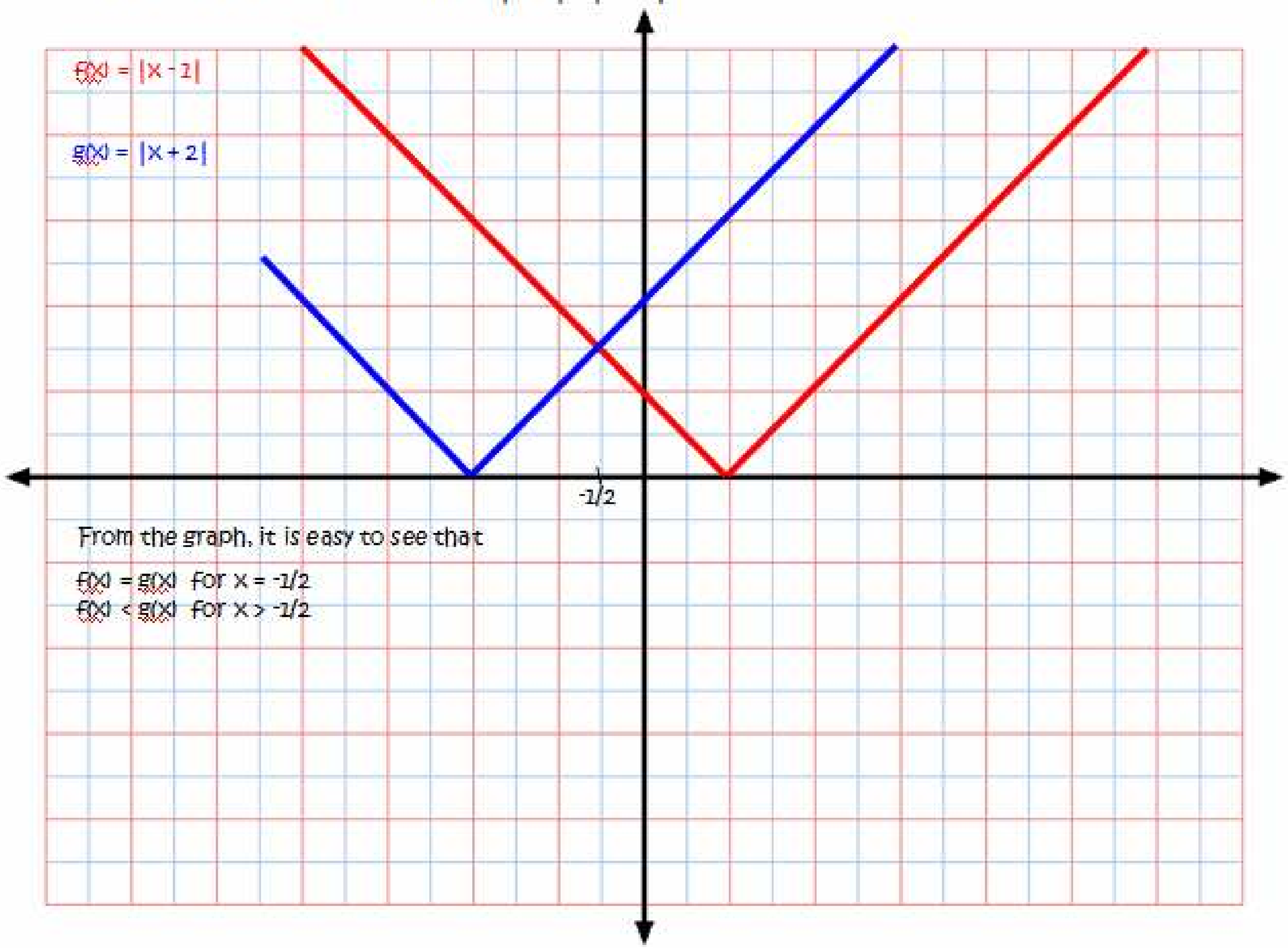
Consider now the intervals (- infinity, -2), [-2, 1] and (1, +infinity)

Below the intervals, write, in one row, the corresponding signs of $x - 1$.
Since $x - 1 > 0$ for $x > 1$, the signs are negative in the intervals (-infinity, -2) and [-2, 1].

In a second row, write the signs of $x + 2$ in these intervals.
Since $x + 2 > 0$ for $x > -2$, the sign is negative only in the interval (-infinity, -2).

VA

Exercise: Find all Values of x for which $|x - 1| < |x + 2|$



Average marks obtained in different groups of students on all questions and on question 3

GROUP (N)	*Average number of questions attempted*	*Average of the total mark in percentages*	*Percent of students who noticed the contradiction in Q.3*
TA-U (6)	**6**	**51**	**33**
PA-U (6)	**6**	**58**	**50**
VA-U (6)	**6**	**75**	**67**
PA-S (10)	**2.8**	**18**	**10**
VA-S (9)	**5.56**	**28**	**22**

	% of those who attempted			
	Q. 2-6		Q.3	
	used			
	l.s.a.	*m.s.a.*	*l.s.a.*	*m.s.a*
TA-U	**24**	**70**	**50**	**50**
PA-U	**27**	**63**	**100**	**0**
VA-U	**5**	**85**	**25**	**75**
PA-S	**20**	**16**	**100**	**0**
VA-S	**33**	**36**	**50**	**50**

l.s.a = less sophisticated approach
m.s.a. = more sophisticated approach

Definitions of l.s.a, & m.s.a.

	l.s.a	m.s.a
Questions 2, 6	Random numerical testing	Systematic numerical testing Graphing Case analysis from definition Case analysis following a procedure
Question 3	Numerical testing Case analysis by definition or procedure	Graphical reasoning Analytical reasoning on signs
Questions 4, 5	Random numerical testing ‘Two-column approach’	Systematic numerical testing Graphing Case analysis Theorem $\lvert x\rvert < a \leftrightarrow -a < x < a$ Complement approach

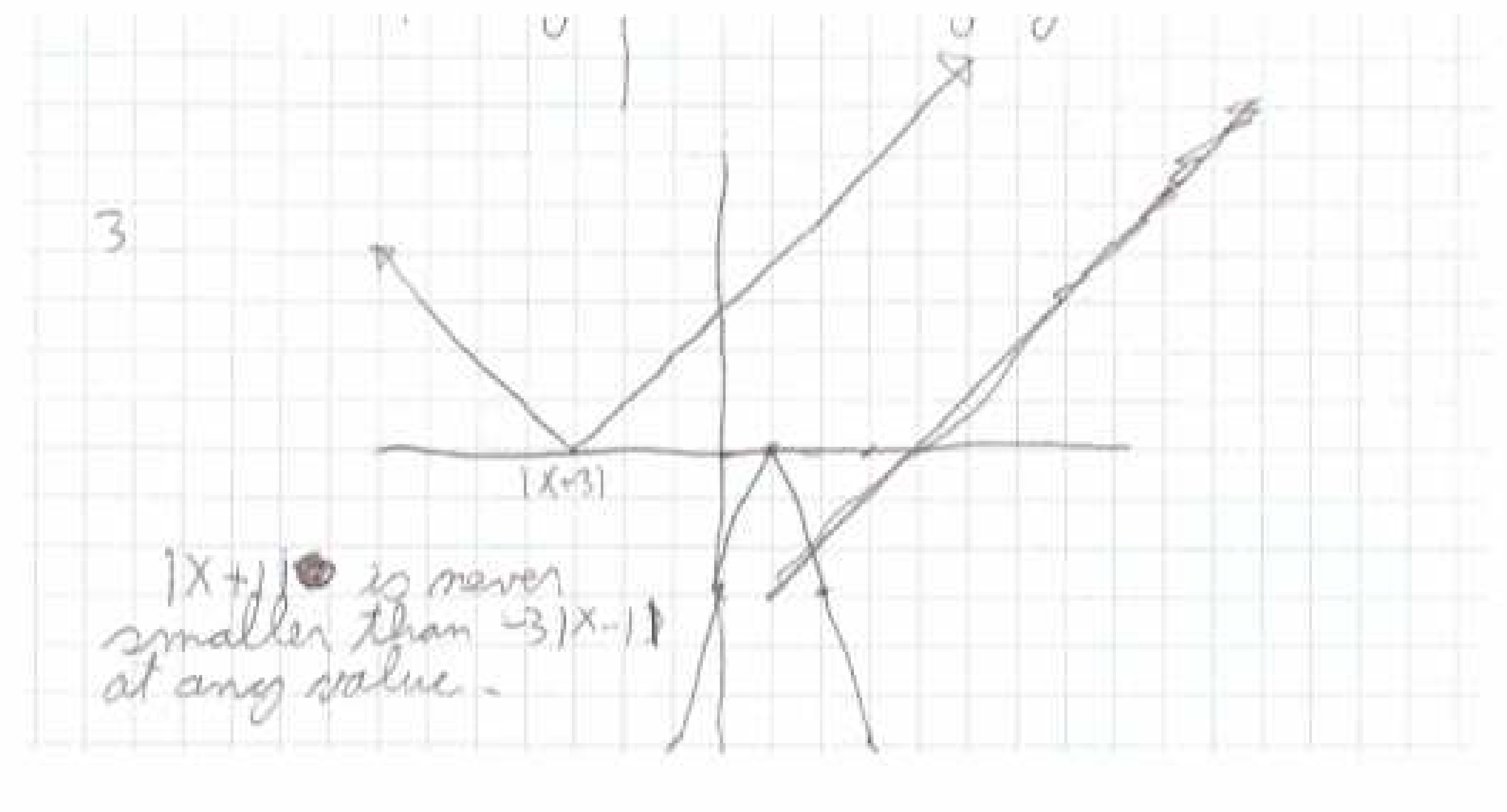


Figure VA-U-2-Ex.3 A fully graphical solution of Exercise 3 by PA-U-2.

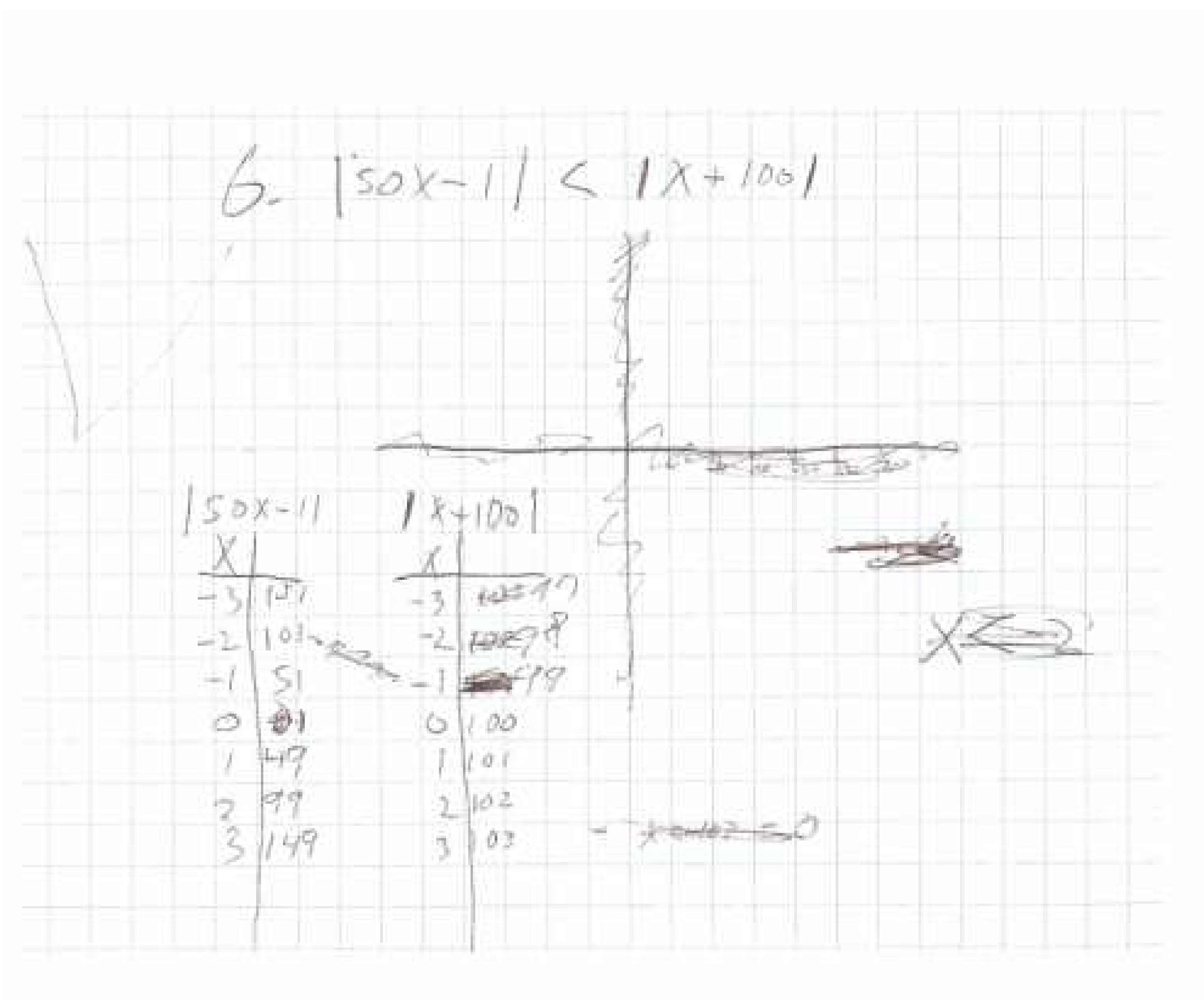


Figure VA-U-2-Ex.6 Systematic numerical testing by VA-U-2 in Exercise 6.

PA-U-6: flawless solutions; an A+ student

- PA-U-6 said that she appreciated the procedural approach to teaching mathematics, where the teacher would explain a method on examples but not dwell too much on the theoretical underpinnings of the method; she found it effective.
- She once had a teacher who would spend time developing the theory in class. She didn’t like it because, ironically, this approach would not leave room for students’ own thinking. All she could do in class was take notes from the board.

Does PA always lead to rote learning?

- The phenomenon of some students' success in procedural approaches has been studied, in particular, by Rittle-Johnson (2006). Her studies suggest that students who **self-explain** are likely to benefit from both direct instruction and discovery learning approaches.
- "Prompts to self-explain promoted learning and transfer equally well under conditions of direct instruction and discovery learning.
- Self-explanation facilitated
  - learning correct procedures to solve novel transfer problems,
  - and retaining these procedures over a delay.
- (…) **Children who did not explain tended to revert to using old, incorrect procedures.**
- In other words, self-explanation strengthened and broadened correct procedures and weakened incorrect procedures, which are central components of improved procedural knowledge (…)
- Contrary to expectations, **self-explanation** during problem solving **did not improve performance on the conceptual knowledge measure."**
- (Rittle-Johnson, 2006).

Discussion

- Why PA prevails in the teaching of mathematics?
- Why is VA better? In what sense is it better?
- What is wrong with TA?