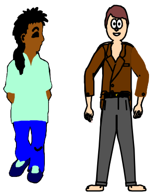
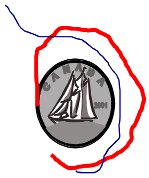
The Procedural Approach (PA) lecture slides

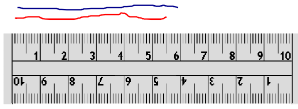
A LESSON ON ABSOLUTE VALUE



Jane and Joe are measuring the circumference of a dime with a string.



Jane' s result is: 55 mm
Joe's result is: 58 mm



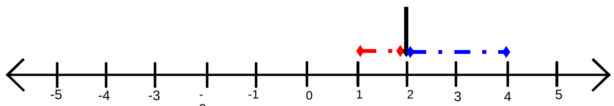
Tom knows the true length of the circumference: 56 mm.
He calculates the difference between the true length and the measurements:
56 - 55 = 1   56 - 58 = -2

He says: Since 1 > -2 then Jane made a bigger mistake than Joe.   Do you agree with Tom?

PA-1

Sometimes we are not interested in knowing whether a measurement was less than or greater than the true value but only in the MAGNITUDE of the difference.

We call this magnitude THE ABSOLUTE VALUE of the number obtained as the difference.



The absolute value of 2 - 1 is equal to 1.
The absolute value of 2 - 4 is equal to 2.

|2 - 4 | = |-2 | = 2 = the opposite of -2 = -(-2)

PA-2

In general, the absolute value of a number x is defined as equal to x for $x > 0$, and equal to $-x$ for $x < 0$.

As for the number zero, we decide that $|0| = 0$.

Symbolically, we usually write this definition as follows:

$$|x| = \begin{cases} x & \text{if } x \geqslant 0 \\ -x & \text{if } x < 0 \end{cases}$$

Thus, the absolute value is never negative; it is positive or zero.

PA-3

We apply this definition to calculate absolute values of concrete numbers, and to simplify expressions with letters involving absolute values.

Here are examples.

Example 1.

To calculate the value of the expression $||-7+5|-|12*2-4||$

$||-7+5|-|12*2-4|| = \quad ||-2|-|20|| \quad = |2-20| \; = |-18| \quad = 18$

Example 2.

To find all numbers x such that $|x-2| < |x+2|$

Solution of Example 2 is on next page

PA-4

Example 2 - continued

We wish to simplify the expression
$|x - 1| < |x + 2|$
so that we are able to identify all numbers x for which this inequality is true.

We can try to guess some numbers for which this inequality is true.

Let's try $x = 5$

$|5 - 1| < |5 + 2|$

$4 < 7$

This is true.
So the inequality $|x - 1| < |x + 2|$ holds for $x = 5$.

How about $x = -3$?

$|-3 -1| < |-3 + 2|$

$|-4| < |-1|$

$4 < 1$

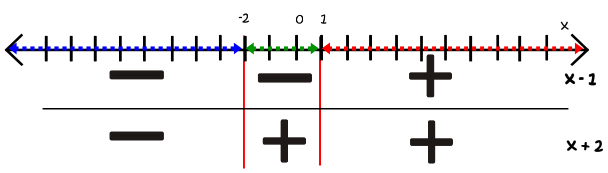
This is false.
So the inequality does not hold for $x = -3$.

PA-5

Example 2 - continued

Since there are infinitely many numbers, it is not possible to find all numbers x for which the inequality $|x - 1| < |x + 2|$ is true just by guessing.
We need a systematic way of finding all these numbers in a finite time.

Here is a method.



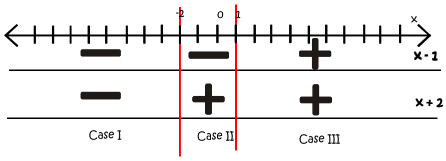
On the number line,
mark the number for which $x - 1 = 0$, i.e. the number 1;
mark also the number for which $x + 2 = 0$, i.e. the number -2.

Consider now the intervals $(-\text{infinity}, -2)$, $[-2, 1]$ and $(1, +\text{infinity})$

Below the intervals, write, in one row, the corresponding signs of $x - 1$.
Since $x - 1 > 0$ for $x > 1$, the signs are negative in the intervals $(-\text{infinity}, -2)$ and $[-2, 1]$.

In a second row, write the signs of $x + 2$ in these intervals.
Since $x + 2 > 0$ for $x > -2$, the sign is negative only in the interval $(-\text{infinity}, -2)$.

PA-6



We have three cases to consider.	The inequality $|x - 1| < |x + 2|$ becomes:	We simplify the inequalities
Case I: $x < -2$	$-(x - 1) < -(x + 2)$	$-x + 1 < -x - 2$
Case II: $-2 \leqslant x \leqslant 1$	$-(x - 1) < +(x + 2)$	$-x + 1 < x + 2$
Case III: $x > 1$	$+(x - 1) < +(x+2)$	$x - 1 < x + 2$

We summarize the cases:

Case I: $x < -2$ and $1 < -2$

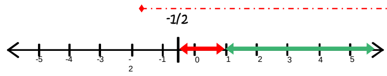
Case II: $-2 \leqslant x \leqslant 1$ and $-1 < 2x$

Case III: $x > 1$ and $-1 < 2$

Case I contains a contradiction, so we reject it. ∅

Case II is equivalent to $-1/2 < x \leqslant 1$

Case III is equivalent to $x > 1$.



So, altogether, the inequality $|x - 1| < |x + 2|$ is equivalent to $x > -1/2$.

PA-7

We concluded that the inequality

$$|x - 1| < |x + 2|$$

is equivalent to $x > -1/2$.

Now let's test our solution.

Let's plug in some number greater than -1/2 back into the initial inequality and see if it's true.

Let's take $x = -1/4$

$|-1/4 - 1| = 5/4$ $\quad$ $|-1/4 + 2| = 7/4$

$$5/4 < 7/4$$

This is true, so our result is probably correct.

PA-8

Exercises

1. Calculate: $||16-24| - |7-56||$

In exercises 2- 6, solve the given inequality

2. $|x -1| < |x + 1|$

3. $|x + 3| < -3|x - 1|$

4. $|2x - 1| < 5$

5. $|2x - 1| > 5$

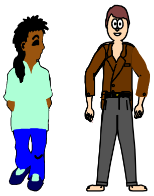
6. $|50x - 1| < |x + 100|$

Thank you for your attention!

PA-9

The Theoretical Approach lecture slides

A LESSON ON ABSOLUTE VALUE



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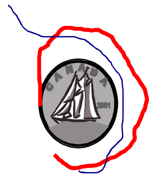

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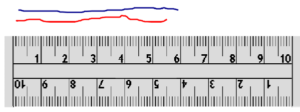
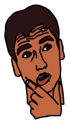




Tom knows the true length of the circumference: 56 mm.
He calculates the difference between the true length
and the measurements:

56 - 55 = 1   56 - 58 = -2

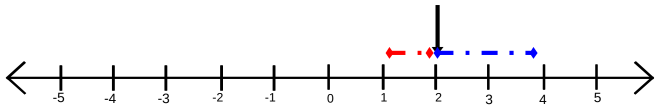
He says: Since 1 > -2 then Jane made a bigger mistake than Joe.

Do you agree with Tom?

TA-1

Sometimes we are not interested in knowing whether a measurement was
less than or greater than the true value
but only in the MAGNITUDE of the difference.

We call this magnitude THE ABSOLUTE VALUE of the number
obtained as the difference.



|2 - 1| < |2 - 4|

The absolute value of 2 - 1 is equal to 1.

$$|2 - 1| = |1| = 1$$

The absolute value of 2 - 4 is equal to 2.

$$|2 - 4| = |-2| = 2 = \text{the opposite of } -2 = -(-2)$$

TA-2

The notion of **absolute value of a number**
is an abstraction from the context of comparing
the magnitudes of measurement errors
which involves calculating the absolute values of differences between
numbers .

But when we write, for example,

$|7|$ or $|-7|$

we really mean,

$|0-(-7)|$ or $|0 - 7|$

both of which are equal to 7.

We can say that the absolute value of 7 is equal to itself,

and the absolute value of -7 is equal to the opposite of itself: -(-7)

TA-3

In general, the absolute value of a number x is equal to x for $x > 0$,
and to -x for $x < 0$.
As for the number zero, we decide that $|0| = 0$.

Symbolically, we usually write this definition as follows:

$$|x| = \begin{cases} x & \text{if } x \text{ positive or } 0 \\ -x & \text{if } x \text{ is negative} \end{cases}$$

Thus, the absolute value of a number is never negative;
it is positive or zero.

TA-4

We apply this definition to calculate absolute values of concrete numbers, and to simplify expressions with letters involving absolute values.

Here are examples.

Example 1.

To calculate the value of the expression $||-7+5|-|12*2-4||$

$||-7+5|-|12*2-4|| = ||-2|-|20|| = |2-20| = |-18| = 18$

Example 2.

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TA-5

Example 2 - continued

We wish to simplify the expression $|x-1| < |x+2|$

so as to identify all numbers x for which this inequality is true.

We can try to guess some numbers for which this inequality is true.

Let's try **x = 5**

$$|5-1| < |5+2|$$
$$4 < 7$$

This is true.
So the inequality $|x-1| < |x+2|$ holds for $x = 5$.

How about **x = -3** ?

$$|-3-1| < |-3+2|$$

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$$4 < 1$$

This is false.
So the inequality does not hold for $x = -3$.

TA-6

Example 2 - continued

Since there are infinitely many numbers, it is not possible to find all numbers x for which the inequality $|x - 1| < |x + 2|$ is true just by guessing.
We need a systematic way of finding all these numbers in a finite time.

Let's reason, using the definition.

$$|x - 1| = \begin{cases} x - 1 & \text{for } x - 1 \geq 0 \\ -(x - 1) & \text{for } x - 1 < 0 \end{cases}$$

There are 2 cases for each absolute value in the inequality...

$$|x + 2| = \begin{cases} x + 2 & \text{for } x + 2 \geq 0 \\ -(x + 2) & \text{for } x + 2 < 0 \end{cases}$$

... so we have four cases altogether to consider.

1. $x-1 \geq 0$ and $x+2 \geq 0$ and $x - 1 < x + 2$
2. $x-1 \geq 0$ and $x+2 < 0$ and $x - 1 < -(x + 2)$
3. $x-1 < 0$ and $x+2 \geq 0$ and $-(x - 1) < x + 2$
4. $x-1 < 0$ and $x+2 < 0$ and $-(x - 1) < -(x + 2)$

TA-7

1. $x-1 \geq 0$ and $x+2 \geq 0$ and $x - 1 < x + 2$
2. $x-1 \geq 0$ and $x+2 < 0$ and $x - 1 < -(x + 2)$
3. $x-1 < 0$ and $x+2 \geq 0$ and $-(x - 1) < x + 2$
4. $x-1 < 0$ and $x+2 < 0$ and $-(x - 1) < -(x + 2)$

We can simplify these conditions:

1. $x \geq 1$ and $x \geq -2$ and $-1 < 2$
2. $x \geq 1$ and $x < -2$ and $x - 1 < -x - 2$
3. $x < 1$ and $x \geq -2$ and $-x + 1 < x + 2$
4. $x < 1$ and $x < -2$ and $-x + 1 < -x - 2$

Simplifying further, we get:

1. $x \geq 1$
2. Contradiction, so we can ignore this case
3. $x < 1$ and $x \geq -2$ and $-1 < 2x$
4. Another contradiction, so we ignore this case, as well

TA-8

We are left with two possible cases

1. $x \geqslant 1$

3. $x < 1$ and $x \geqslant -2$ and $-1 < 2x$

Condition 3 is equivalent to $-2 \leqslant x < 1$ and $-1/2 < x$

Which simplifies to $-1/2 < x < 1$

So, altogether, $-1/2 < x < 1$ or $1 \leqslant x$

This can be combined into a single condition:

$$-1/2 < x$$

TA-9

We concluded that the inequality

$$|x - 1| < |x + 2| \qquad (*)$$

is equivalent to $x > -1/2$.

We proceeded logically from the definition.
So, if we didn't make any computational mistakes, this should be correct. But computational mistakes are always possible.

Let us test the result by plugging some numbers greater and smaller than -1/2 back into the initial inequality.

Let's take $x = -1/4$

$|-1/4 - 1| = 5/4$ $|-1/4 + 2| = 7/4$

(*) becomes $5/4 < 7/4$

This is true,
as predicted by our solution.

Let's take $x = -1$

$|-1 - 1| = 2$ $|-1 + 2| = 1$

(*) becomes $2 < 1$

This is false,
as predicted by our solution.

The compatibility of these examples with our solution increases our confidence that no computational mistakes have been made.

TA-10

Exercises

1. Calculate: $||16-24| - |7-56||$

In each of the following exercises, find all values of x for which the given inequality is true

2. $|x - 1| < |x + 1|$

3. $|x + 3| < -3|x - 1|$

4. $|2x - 1| < 5$

5. $|2x - 1| > 5$

6. $|50x - 1| < |x + 100|$

Enjoy the challenge!

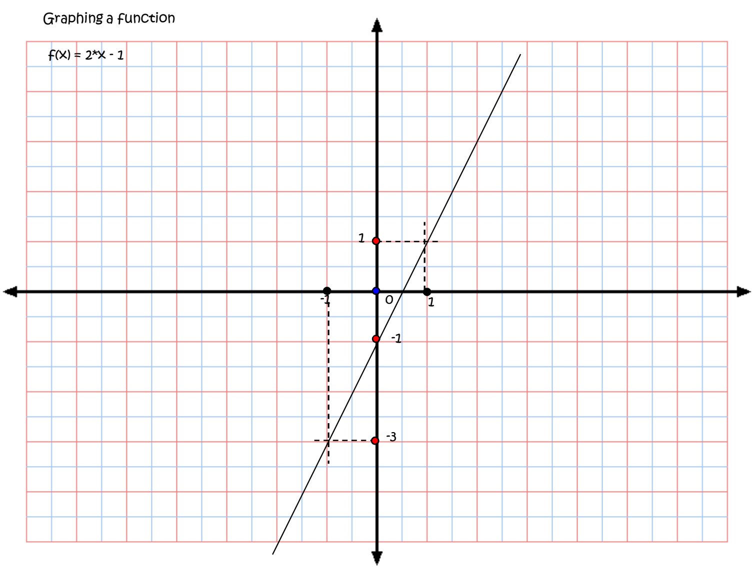
Thank you for your attention.

The Visual Approach lecture slides

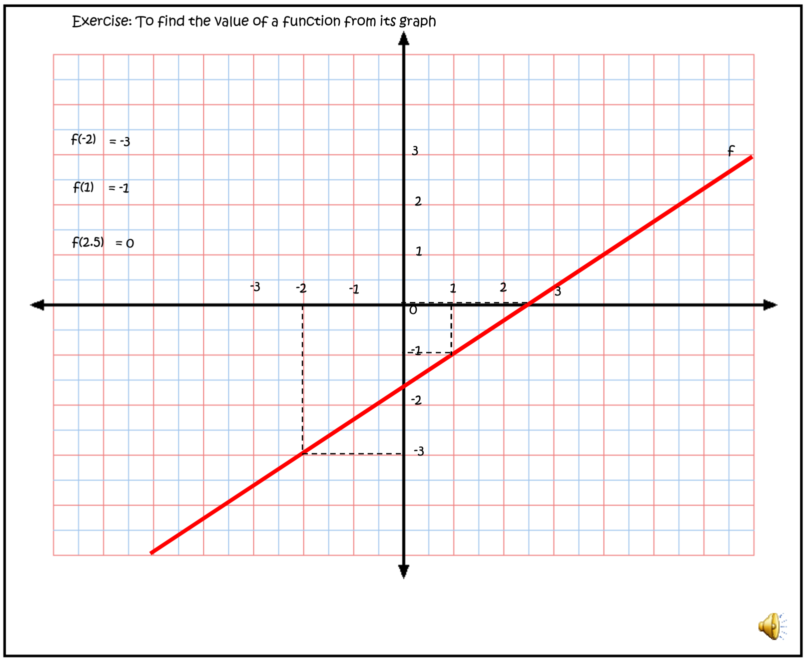
Solving inequalities with absolute values

- We start by the idea of using graphing to compare functions
- Next, we introduce the concept of absolute value
- And, finally, we learn how to solve inequalities with absolute value

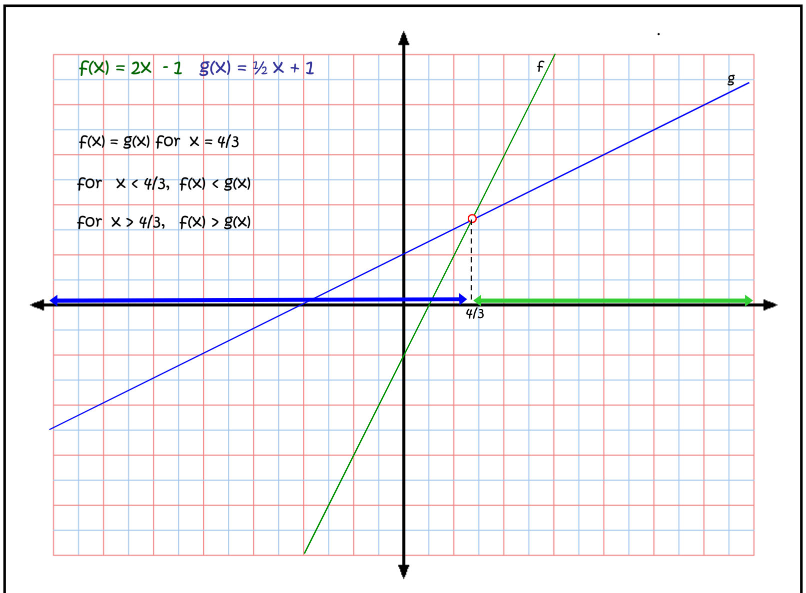
VA-1



VA-2



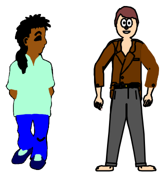
VA-3



Comparing the values of two functions, by looking at their graphs

VA-4

A LESSON ON ABSOLUTE VALUE



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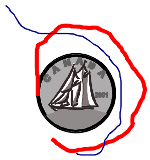

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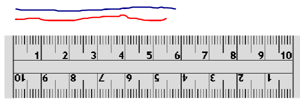

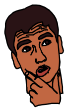

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VA-5

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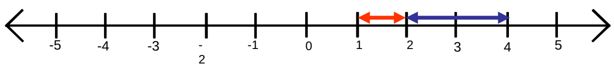


|2 - 1 | < |2 - 4|

The absolute value of the difference 2 - 1 is equal to 1.

|2 - 1 | = | 1 | = 1

The absolute value of the difference 2 - 4 is equal to 2.

|2 - 4 | = |-2 | = 2

2 = the opposite of -2 = -(-2)

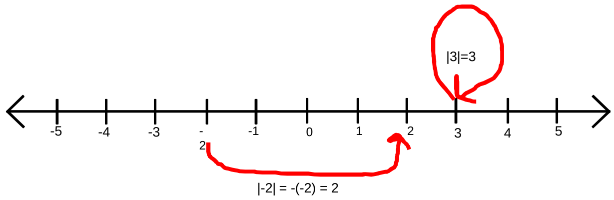
VA-6

In general, we define the absolute value of a number *x* as equal to *x* for positive numbers *x*, and equal to the *opposite of x* for negative numbers *x*.
As for the number zero, we decide that |0| = 0.
The absolute value of a number is thus never negative.

We can think of the absolute value of a number as a function, defined as follows:

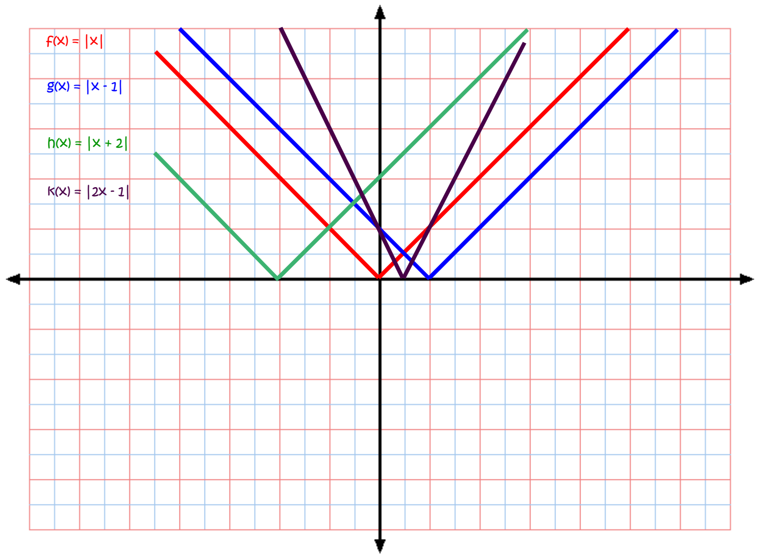
$$|x| = \begin{cases} x & \text{if } x \text{ is positive or 0} \\ -x & \text{if } x \text{ is negative} \end{cases}$$

Here is a visual representation of this definition, using the number line:



The absolute value of a positive number is equal to itself.
The absolute value of a negative number is equal to its opposite.

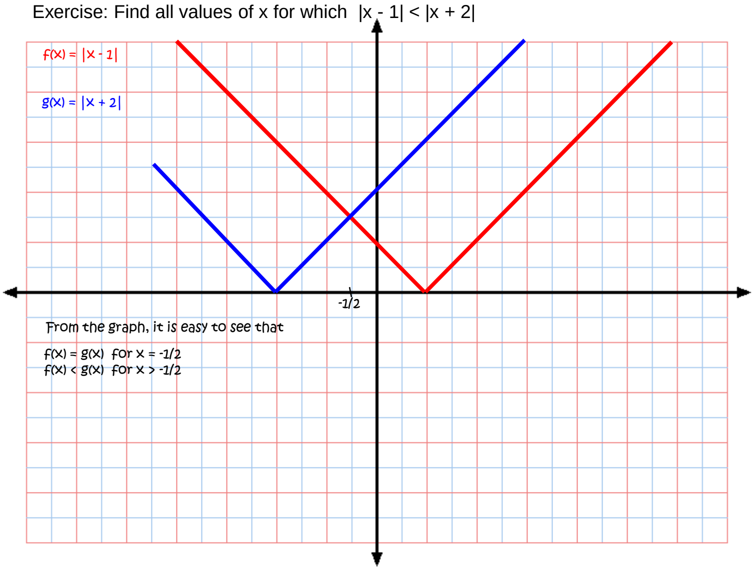
VA-7



Graphing absolute value functions

VA-8

Exercise: Find all values of x for which |x - 1| < |x + 2|



VA-9

Suppose it is not so easy to see from the graph
for what values of x the values of one function are less than the values of the other one.

What does one do in such cases?

We can use reasoning and algebra.

Let's look again at the inequality | x - 1 | < |x + 2|.

(a) |x - 1 | = x - 1 for x - 1 ≥ 0
(b) |x - 1| = -(x - 1) for x - 1 < 0

(c) |x + 2| = x + 2 for x + 2 ≥ 0
(d) |x + 2| = -(x + 2) for x + 2 < 0

So we have 4 cases to consider:

(a) & (c): for x ≥ 1 & x ≥ -2, the inequality is x – 1 < x + 2

(a) & (d): for x ≥ 1 & x < -2, the inequality is x – 1 < -(x + 2)

(b) & (c): for x < 1 & x ≥ -2, the inequality is –(x – 1) < x + 2

(b) & (d): for x < 1 & x < -2, the inequality is –(x – 1) < -(x + 2)

VA-10

Solving the inequality | x - 1 | < |x + 2|, continued

(a) & (c): for x ≥ 1 & x ≥ -2, the inequality is x – 1 < x + 2

(a) & (d): for x ≥ 1 & x < -2, the inequality is x – 1 < -(x + 2)

(b) & (c): for x < 1 & x ≥ -2, the inequality is –(x – 1) < x + 2

(b) & (d): for x < 1 & x < -2, the inequality is –(x – 1) < - (x + 2)

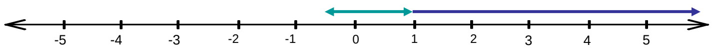
Simplify

(a) & (c): for x ≥ 1, the inequality is – 1 < 2

(a) & (d) contradiction

(b) & (c): for x < 1 & x ≥ -2, the inequality is x > -1/2

(b) & (d) contradiction



x > -1/2

VA-11

Exercises

1. Calculate: **||16-24| - |7-56||**

In each of the exercises below, find the values of the number x for which the given inequality is true

2. |x -1 | < |x + 1|

3. |x + 3| < -3|x - 1|

4. |2x - 1| < 5

5. |2x - 1| > 5

6. |50x - 1| < |x + 100|

Thank you for your attention!

VA-12