CONCORDIA UNIVERSITY

A STUDY OF RELATIONSHIPS BETWEEN THEORETICAL THINKING AND HIGH ACHIEVEMENT IN LINEAR ALGEBRA

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SUMMARY

Few would argue against the statement that theoretical thinking is, in principle, necessary for the learning of linear algebra at the university level. But is it also necessary *de facto*, i.e. for successfully completing a linear algebra course? Our study started with this question. In our research we encountered many theoretical and methodological problems. We needed to make it explicit what we meant by theoretical thinking in order to explain why we considered this mode of thinking necessary for the learning of linear algebra. This led us to proposing a model of theoretical thinking, which then needed to be operationalized in terms of concrete mathematical behaviors. Moreover, we needed to define some measures of a student's or a group of students' disposition to theoretical thinking in order to compare it with the students' grades in the courses.

We developed our theory and methodology along with our study of data from interviews with 14 students who achieved high grades in a first linear algebra and some of whom obtained high grades also in a second linear algebra course. We also studied the final examination questions in these courses from the point of view of the level of theoretical thinking that was, in principle, needed to obtain full marks.

The report has five chapters and is preceded by an introduction, where the research question is formulated and put in context.

In Chapter I we describe our model of theoretical thinking. We argue that practical and theoretical thinking are deeply intertwined in the development of scientific knowledge, and we posit that theoretical thinking develops *against* practical thinking, which thereby also constitutes a condition for the existence and significance of theoretical thinking. We then justify why we think that theoretical thinking is necessary for understanding linear algebra. The chapter concludes with a summary of the postulated features of theoretical thinking. We name the main categories of these features "reflective", "systemic" and "analytic" thinking.

In Chapter II we present our interviews with a group of 14 students who were selected for having achieved high grades in a section of a first linear algebra course (they were all taught by the same instructor). Six of these students achieved high grades also in a second linear algebra course. We explain our design of the seven interview questions and our method of coding students' responses: the code [1, 0] is used to represent theoretical behavior, [0, 1] represents practical behavior and [1, 1] represents a mixture of theoretical and practical

behavior. Six questions in the interview had mathematical content and the seventh question was meant to incite the students to reveal their beliefs about, among others, truth, mathematical knowledge, and reasons for taking mathematics courses.

In describing students' behavior in each question we explain how we related this behavior with our model of theoretical thinking. This leads us to an operationalization of the model in terms of features of students' behavior; we identify 18 "theoretical behavior" features

In Chapter III we present the contents of the linear algebra courses taken by the interviewed students and we analyze in detail the final examinations given in these courses. We look at the examination questions from the point of view of the relevance of theoretical thinking in their solution. We point to ways of reformulating the questions so that theoretical thinking becomes more relevant in their solution.

Chapter IV contains a quantitative evaluation of the students' theoretical behavior. We treat the numerical coding introduced in Chapter II as scores and we define three categories of statistical indices as functions of these scores: expectation, tendency and capability, which we apply to the group as a whole (group indices) and to individual students (individual indices). We use these indices to examine the relations between the students' disposition in the sense of expectation, tendency and capability, and their grades in the courses. We also apply the group indices to dress a profile of theoretical thinking strengths and weaknesses in the whole group of students, compared with the profile of the subgroup of students who achieved high grades in both courses. These profiles are formulated in terms of rankings of the 18 theoretical behavior features according to the indices of expectation, tendency and capability. We conclude with a comparison of the students' disposition to theoretical thinking and the relevance of this kind of thinking in solving the examination questions.

In Chapter V we formulate some more general conclusions from the study and, looking back at our research methodology, we reflect on the empirical value of our findings.

The report closes with some remarks on the theme of the idea of a university. We propose that the development of theoretical thinking is an important part of the mission of the university and that there is much room for improvement in mathematics courses in contributing to fulfill this mission.

TABLE OF CONTENTS

INTRODUCTION

The aims of the study

After many years of trying to understand students' difficulties in the learning of linear algebra at the undergraduate level, we came to the conclusion that all our detailed and elaborate explanations could perhaps be summarized by saying that one of the main reasons, why students' understanding of linear algebra departs in many ways from the theory, is that students approach it with a practical rather than a theoretical mind (Sierpinska, 2000).

This statement sounds quite obvious, almost tautological. Of course, linear algebra, with its axiomatic definitions of vector space and linear transformation, is a highly theoretical knowledge, and its learning cannot be reduced to practicing and mastering a set of computational procedures. Practical thinking could perhaps be sufficient for studying introductory level "vectors and matrices" courses, focused on methods of solving linear equations and vector and matrix arithmetic, but not for an advanced linear algebra course. But as soon as we tried to formulate what exactly we meant by "theoretical" (as opposed to "practical") thinking, what are its specific features, and in what way these specific features are relevant for the learning of linear algebra, we realized that the above statement is not so clear.

Moreover, we realized that our statement is concerned with the relevance of theoretical thinking for the learning of linear algebra. It may be a very different matter to ask if theoretical thinking is relevant for the students' success in a linear algebra course. This is an important pedagogical question because, if linear algebra cannot be understood well enough without theoretical thinking, but theoretical thinking is not necessary for success in a given linear algebra course, then what is the point of making this course compulsory for so many students?

These questions were at the start of our research, which then focused on making it clear to ourselves what we meant by theoretical as opposed to practical thinking, by

(a) elaborating a model of "theoretical thinking" and,

(b) developing a method of evaluating an individual's or a group's disposition[1] to thinking theoretically in the sense of the postulated model.

Task (b) required an operationalization of the model of theoretical thinking in terms of

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[1] We use this word in the sense of Resnick (1987, p.41): "The term disposition should not be taken to imply a biological or inherited trait. As used here, it is more akin to a habit of thought, one that can be learned, and, therefore, taught".

features of mathematical behavior assumed as corresponding to the various aspects of the model (qualitative operationalization), and some numerical indices or measures of students' disposition to theoretical thinking (quantitative operationalization).

This methodology was developed and probed in interaction with the study of three sources of information.

Source 1: Existing literature on different kinds of knowing in philosophy, psychology and mathematics education. This source provided us with conceptual categories useful in the construction of our model of theoretical thinking, and in justifying why, a priori, theoretical thinking is relevant in learning linear algebra.

Source 2: Interviews with 14 students, who achieved high grades (‡ 80%) in a first linear algebra course. Six of these students achieved high grades also in a second linear algebra course. Two other students did not take the second linear algebra course (one switched from mathematics to psychology and the other was doing her internship in a business company in the frame of the co-op program). Of the remaining six students, three obtained grade B (70-79%), two - grade C (60%-63%), and one - grade D († 50%). Both courses were taught by the same instructor, who also set the course syllabus, chose the textbook and wrote the final examinations. The questions in the interviews aimed at probing students' disposition to theoretical thinking in mathematical contexts. We were interested in knowing if this disposition was necessarily strong in these high achievers: was it possible to be more inclined to thinking in practical ways yet achieve high grades in the two linear algebra courses?

Source 3: The final examination questions in the two courses taken by the students we interviewed. An analysis of these questions was expected to explain a possible positive answer to the last question above: perhaps very little theoretical thinking was necessary to obtain full marks on the examinations problems. While the interviews could give us an idea of the potential for theoretical thinking in the group of high achievers, analysis of the examination questions could indicate the minimal level of theoretical thinking necessary to succeed in the course. The difference between these two levels could be interpreted as a measure of the wasted students' intellectual possibilities.

Our curiosity about the relations between high achievement and theoretical thinking notwithstanding, methodological questions often overshadowed other concerns in our research. The construction of a model of theoretical thinking, the definition of the corresponding theoretical behaviors and of their measures were taking much of our thought and effort, and we consider these as important results of our study. The empirical results were restricted to a particular group of 14 students who achieved high grades in specific circumstances. From the point of view of empirical results, our research should be considered as a case study. As any case study, it has cognitive value to the extent it provokes the reader to reflect on similar or contrasting cases and affords him or her some analytic tools for their analysis.

Research areas intersecting with our study

Relations between ways of knowing and academic success

The question of relations between students' ways of knowing and their academic success has interested many researchers. In social studies there are the well known large sample, longitudinal studies of Perry (1970), Belenky, Clinchy, Goldberger, and Tarule (1997), Baxter Magolda (1992), Schommer, Carvert, Gariglietti and Bujaj (1997), to mention but a few. These studies have focused mainly on students' views about truth, scientific procedures and methods of validation, i.e. on students' epistemological positions. In mathematics education research, epistemological beliefs of students and teachers have also occupied an important place. In particular, drawing on Baxter Magolda's research, Weinstein (1998) has developed a framework for the study of mathematics students' beliefs about the epistemology, learning and pedagogy of mathematics. Some mathematics educators have tried to identify not only such "meta-mathematical" beliefs but also features of mathematical behavior that could discriminate between "experts" and "novices". The most frequently cited author in this domain is probably Schoenfeld (e.g., 1989).

Recently, Carlson (1999) observed six successful graduate mathematics students as they solved mathematical problems and she also questioned them about their "meta-mathematical" views. Among other things, the students were asked to express their opinions about the conditions of learning of mathematics (e.g., they were asked to evaluate the level of their endorsement of statements such as, "achievement depends more on persistent effort than on the influence of a teacher or a textbook"), and the necessary conditions of mathematical understanding (e.g., "reconstructing new knowledge in one's own way instead of just memorizing it as given"). One of Carlson's conclusions was that non-cognitive factors, such as persistence in the study of mathematics and having had a particularly skillful and devoted mentor, play an important part in a student's mathematical success. Concerning the observed students' mathematical problem solving behavior, the most striking feature for Carlson was that, although these students exhibited weak control decisions and were rather ineffective in using recently learned knowledge, they "were carefulÉto offer only responses that appeared to have a logical foundation" (Carlson, ibid. p. 255). For us, this trait of behavior would be considered as corresponding to an aspect of theoretical thinking.

Distinction between theoretical and practical thinking

In mathematics education

In a way, mathematics education has always been concerned with the opposition between practical and theoretical thinking, keeping the balance between the two apparently being the hardest problem to solve. Sometimes teaching was considered to be too theoretical and there were calls to permeate the teaching of mathematics with references to the real life context, the specific culture of the students, and their spontaneous ways of thinking. At other times, mathematics educators were concerned that teaching mathematics "in contexts" may compromise the essentially theoretical character of mathematical knowledge. In particular, the work of Steinbring afforded many examples of ways in which the theoretical character of mathematical knowledge could be lost in an approach based on the methodical principle of "ascent from the concrete to the general" (e.g. Steinbring, 1991; Seeger and Steinbring, 1992). There have been attempts to achieve a balance between the two modes of thinking, with the idea that, while there is no mathematical thinking without theoretical thinking, the latter must be grounded in a reflection on a practice. The practice may be related to modeling and application problems as well as solving theoretical problems. This idea of balance between theoretical and practical thinking in mathematics was strongly promoted by Freudenthal, for whom mathematics was a human activity, namely an activity of mathematizing, both of subject matter from reality and of mathematical subject matter (Freudenthal, 1973; Gravemeijer, 1997). Our own reflection about a possible such balance in the teaching and learning of linear algebra was started in (Sierpinska, 1995) and continues in the present work.

Theoretical thinking has not always been discussed in explicit terms in mathematics education. Explicit use of the term "theoretical knowledge" (rather than "theoretical thinking") is found in the above mentioned works of Steinbring. Recently, "theoretical knowledge" has started to be an object of study by a group of Italian researchers led by Paolo Boero. Their attention to this subject could perhaps be explained by their interest in epistemologies that underscore the social and cultural roots of knowledge (as suggested in the cognitive developmental theory of Vygotsky, 1987). In their 1997 paper, Boero, Pedemonte and Robotti were saying that neither the commonly used educational strategies, nor those promoted by constructivists, have been able to find a solution to the problem of bridging the gap between "the expressive forms of students' everyday knowledge and the expressive forms of theoretical knowledge, between the students' spontaneous ways of getting knowledge through facts, and theoretical deduction, and between students' intuitions, and the counter-intuitive content of some theories". Boero and his collaborators have been working on ways of approaching theoretical knowledge in teaching mathematics in a series of teaching experiments designed along a didactic strategy called the "voices and echoes games" (Boero, Pedemonte, Robotti and Chiappini, 1998; Garuti, Boero and Chiappini, 1999).

In mathematics education literature, there are many distinctions similar to that between theoretical and practical thinking without an explicit use of these terms. In fact, such distinctions have been present in all theories of learning mathematics used or developed in mathematics education research. This opposition was usually seen as dialectic, in the sense that the construction of mathematical knowledge involved an interaction between two conflicting modes of thinking, one geared towards immediate action, the other - towards reflection on this action and its conceptual structuring. Sometimes the source of inspiration of these theories was the Piagetian concept of reflective abstraction. We find this basic practical/theoretical dialectic in Brousseau's theory of didactic situations of action, formulation and validation (Brousseau, 1997). We find it in Sfard's notion of the dual operational-structural nature of mathematical concepts (Sfard, 1992). The dialectic is also in the APOS theory, which proposes a mechanism for constructing mathematical knowledge (Dubinsky, 1997). This theory identifies several stages or components of theorization: interiorization of actions leading to the construction of processes; encapsulation of operations on processes leading to objects, which can, in turn, become objects of new actions. The theory of "procepts" (Gray and Tall, 1994) is also grounded in the fundamental dialectic between "doing" and "reflecting". More recently, this latter theory has been extended to discuss success and failure in mathematics (Gray, Pinto, Pitta, Tall, 1999; Tall, Gray, Bin Ali, Crowley, DeMarois, McGowen, Pitta, Pinto, Thomas, Yusof, 2001). Our study has some affinity with the work of these researchers, for bringing together the issues of academic success and theoretical thinking.

In philosophy

In philosophy, distinctions between a practical way of thinking, aimed at immediate action, and a theoretical way of thinking, based on reflection, have been made since Antiquity. Aristotelian categories of empirical knowledge, art, craft, productive science and theoretical science have long shaped Western epistemological reflection. Hume's notion of "impressions of reflection", Kant's notion of "practical reason", Dewey's idea of "reflective thinking", Cassirer's distinction between substantial and relational thinking are all akin to what we would wish to understand by theoretical thinking. Although our terminology (theoretical/practical) is perhaps the closest to Kant's, our notion of "practical thinking" is different from this philosopher's notion of "practical reason", because we do not make the distinction on the basis of involvement or non-involvement of will in a judgment. For us, there is no such thing as "pure reason". All thinking and thus also theoretical thinking is to some extent "impure" or related to will, emotions and desires. The difference between a theoretically thinking individual and a practically thinking individual is that the former is able to distance him or herself from what he or she wills and feels, and engage in hypothetical thinking: "What if I felt differently about this matter? What would be the logical consequences of this approach?" The practical thinker makes one with his or her will and emotions and is unable or reluctant to looking at them from this perspective. We also differ from Kant in our attitude towards "technical mathematical knowledge". Kant excluded "technical reason" from "practical reason" claiming that technical thinking belongs to "theoretical reason":

> Propositions, which in mathematics or physics are called practical, ought properly to be called technical. For they have nothing to do with the determination of the will; they only just point out how a certain effect is to be produced, and are therefore just as theoretical as any propositions which express the connection of a cause with an effect.
>
> (Kant, 1873/1948, p.113)

We do not classify technical mathematical knowledge (formulas, algorithms) as belonging to either the practical or the theoretical domain, because we are concerned with processes of thinking and not with propositions disconnected, as it were, from the thinking individual. For us, all depends on whether the individual is only applying a formula or algorithm to solve a particular question, or reflecting on it and verifying if it can, indeed, be applied. In the former case the person would be thinking practically, in the latter - theoretically.

Cognitive psychology has made many fine distinctions among ways of thinking and knowing, but our understanding of theoretical as opposed to practical thinking was inspired mainly by the Vygotskian distinction between scientific and everyday concepts (Vygotsky, 1987).

Outline of the report

In Chapter I we describe our model of theoretical thinking and we justify why we think that theoretical thinking is necessary for understanding linear algebra. We explain how the model was inspired by Vygotsky's distinction between scientific and everyday concepts and in what sense our epistemological assumptions are different from those of Davydov's interpretation of Vygotsky's theory. We also argue that practical and theoretical thinking are deeply intertwined in the development of scientific knowledge. We posit that theoretical thinking develops *against* practical thinking, which thereby also constitutes a condition for its existence and significance. We conclude the chapter with a summary of the postulated features of theoretical thinking. We name the main categories of these features "reflective", "systemic" and "analytic" thinking.

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Six questions in the interview had mathematical content. The first required a classification of a few algebraic expressions. The second and third were related to the concept of linear dependence/independence. The fourth question asked the students to decide which of two graphs, given in log-log scales, represented a linear function. The fifth question introduced a concept of "brillig number" (odd prime + 2) and asked students some questions about this concept (in particular, the students were asked to pick a definition from among several statements about this concept). The sixth question dealt with the notion of one-sided neutral element in a finite algebraic system with one operation. The seventh question provoked the students to discuss their beliefs about, among others, truth, mathematical knowledge, and reasons for taking mathematics courses. In describing students' behavior in each question we explain how we related this behavior with our model of theoretical thinking.

In particular, in discussing the third question we give an illustration of how theoretical thinking can be activated by an unexpected outcome of practical thinking. This analysis leads us to an operationalization of the model in terms of features of students' behavior; we identify 18 "theoretical behavior" features.

In Chapter III we present the contents of the linear algebra courses taken by the interviewed students and we analyze in detail the final examinations given in these courses. We look at the examination questions from the point of view of the relevance of theoretical thinking in their solution. We point to ways of reformulating the questions so that theoretical thinking becomes more relevant in their solution.

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CHAPTER I. A MODEL OF THEORETICAL THINKING

In this chapter we first describe a general epistemological perspective from which we look at mathematical thinking. We then postulate a definition of theoretical thinking in terms of a certain number of more detailed properties, and we argue for the relevance of theoretical thinking in understanding linear algebra.

SOME BASIC EPISTEMOLOGICAL ASSUMPTIONS

The Vygotskian inspiration and focus on the individual thinker in a civil society

Boero and his collaborators' definition of "theoretical knowledge" was inspired by Vygotsky's characterization of scientific concepts as (a) constituted into conceptual systems expressed by statements and symbolic notations, and (b) formed in the mind on the basis of concretization from general statements (while spontaneous concepts are formed on the basis of generalization and verbalization from concrete experience). Boero et al. also took from Vygotsky the assumption that theoretical thinking does not develop spontaneously in children as one last "stage" of their cognitive development, but requires special nurturing.

We also accepted these assumptions, but we used them to construct a model of theoretical *thinking*, not theoretical *knowledge*. This implies a different perspective. Boero et al. were interested in engaging students with various forms of theoretical "knowledge" as transmitted by a culture (through written documents, for example) and as re-constructed, through specially designed tasks and discussions, in their classrooms. This point of view is closer to Leont'ev's (1959) evolutionary psychology and Davydov's (1990) "philosophical pedagogy" (sometimes referred to as "Activity theory"), than Vygotsky's psycho-pedagogy, in that Leont'ev and Davydov give priority to society over the individual.

Vygotsky, who was at the source of a long tradition in Soviet Marxist psychology and pedagogy, was himself still focused on the individual. The individual was also in the center of our attention. We saw culture and society not as "factors" that *influence* the development of the individual but as an environment with which the individual *interacts*. This environment acts on the individual as much as the individual acts on the environment. Society, we assumed, is an emergent of the many interactions between participants; it is not, as in the dialectical materialist philosophy endorsed by Davydov, a super-entity, which "*confersÉthe* *historically developed forms of Éactivity"* on the individual (Davydov, ibid. p. 232). Referring to Marx and Lenin, Davydov proposed that, as far as theory of cognition is concerned,

> [I]t is necessary to elucidate the *formation* of different forms of thought, which is regarded in the dialectical materialist theory of cognition as an 'objective process in the endeavors of humanity, a functioning of human civilization, of society as the real subject of thought' [reference to Lenin]. An individual person's thought is the functioning of historically developed forms of society's activity, which have been *conferred* on him. One of the basic weaknesses in traditional child and educational psychology has been that it has not treated the individual's thought as a historically developed function of its 'real subject', a function that is learned by him. [Emphases in the original].
>
> (Davydov, p. 232)

Dialectical materialism positioned itself as far as possible from the actual, individual forms of knowing and thinking, substituting logic of thought for psychology of thinking and postulating to explain the ontogenesis of concepts by a rational reconstruction of their phylogenesis. We did not endorse this point of view on the status of society and its relation to the individual, and, especially, we did not endorse the postulate to replace "civil society" by "human society" (ibid. p. 243).

The civil society point of view played an important role in our understanding of theoretical thinking. We wanted to stress the individual theoretical thinker's attitude of *independent* and *critical* participation in the construction of knowledge, which would allow him to *overcome* the intellectual habits "conferred on him" by the "historically developed forms of society's activity".

Mathematical theoretical thinking is in opposition with material labor

We also rejected another assumption of dialectical materialism, namely, that all forms of thought (including theoretical cognition) are a necessary historical consequence of labor and material production.

In dialectical materialism labor was defined as a productive activity, which is concerned with practical objects; its aim was taken to be the modification of nature (Davydov, p. 232, in reference to Engels). Since man is not a purely natural being but a social being, a member of the human society, man's production has a "universal character" which is absent from the productive activity of an animal (Davydov, p. 236-7, in reference to Marx and Engels). In order to modify nature, man must be able to re-produce natural events. But because his production has the universal character just mentioned, the re-production of a natural event implies an understanding of the conditions of its happening, and leads to the identification (in the sense of discovery) of the laws and mechanisms of nature. This engages theoretical cognition.

In Leont'ev's (1959) account of the evolutionary process of the emergence of theoretical thinking, the moment of the alienation of speech from its communicative function was stressed.

> The development of speech does not start with conversations on arbitrary subjects. Its function is first determined by the inclusion of speech in the collective activity of people.ÉThe new step consists in the separation of the theoretical, cognitive function of speech from the communicative function of speech. This separation comes into being in the next historical stage. Its historical premise is the separation of the function of organization of production and exchange [of goods], and, as a consequence, of the function of management. These circumstances award speech an autonomous motivation, i.e. transform it into an autonomous activity. This verbal activity serves not only communication but also theoretical aimsÉ (Leont'ev, 1959, p. 240; my translation from Russian, AS).

Both Davydov and Leont'ev referred to large scale evolutionary processes. They were proposing a "rational reconstruction" of the historical emergence of theoretical thinking, based on the philosophical foundations of dialectical materialism. Their reasoning was suggesting that the development of theoretical thinking was inevitable, that it was a "historical necessity". We have some reservations with respect to this assumption. Following Bachelard (1938), we prefer to assume that not only theoretical thinking was not a "natural" consequence of practical thinking but that it developed *against* such thinking. For us, theoretical thinking could only occur to people who decided to stop trying to modify nature for a while and engaged, instead, in questioning this action and its underlying beliefs and assumptions. We think that mathematics has evolved because some individuals started seeing general patterns in practical activities and started developing sophisticated ways of representing these patterns. We agree with Leont'ev on this particular point: that in theoretical thinking language has an autonomous status as an object of creation and reflection.

Relations between social and epistemological meanings

In relation to the epistemological question of the sources and nature of meaning, we assume that in the domain of practical thinking the so-called "social meanings" prevail, while in the theoretical domain there is an effort to work with "epistemological meanings" (McHugh, 1968).

Social meanings are created on the basis of utterances produced in social situations. They are obtained through an (often tacit) "contract of agreement" among the participants according to which some events are then considered "real" and other "illusory", some conclusions - plausible and other - rejected as questionable or fantastic, and some things are "right" while other are "wrong" to say or do.

In the theoretical domain the ambition is to decontextualize, depersonalize and detemporalize statements, and to construct their meanings as meanings of propositions embedded within a system of propositions. In the system, the "contract of agreement" regarding the validity of statements and the coherence of the system as a whole is assumed to be made explicit in terms of basic assumptions and rules of inference. In this domain, decisions are assumed to be based on an emotionless analysis of all logically possible cases. The awareness of assumptions turns all statements in the theoretical domain into conditional statements. In this domain, nothing is "taken for a fact"; on the other hand, "facts" form the backbone of thinking in the practical domain.

Theoretical thinking is not, of course, very "practical" in the common sense of the word: theoretical thinking is not more economical in terms of time and more effective in terms of the success of actions that would be undertaken on its basis. A strictly theoretical thinker would be a pathology. Damasio (1994) described the case of "Elliot", who became such purely theoretical thinker after the surgical removal of a brain tumor. His demeanor was unemotional but perfectly logical. He could systematically classify documents and consider all possible courses of action but he was unable to make up his mind and choose to pursue one such course of action. He would rather endlessly discuss the possible criteria for sorting documents or choosing a course of action

Therefore, the distinction between social and epistemological meanings can only be methodological. "Epistemological meanings" do not exist in the reality of scientific cultures in a pure form. Social and epistemological meanings are difficult to separate because the latter enjoy existence only to the extent that this existence is socially recognized (through publications, for example). The epistemological meaning of a statement is relative to the particular culture (or institution) that produced it and to that within which it is interpreted (Chevallard, 1992).

In particular, as any mathematical concept has had a history and has been part of a culture, its meaning is not confined to a formal interpretation of its definition in a presently accepted academic monograph. There are normally several meanings, some of them more general than other, partly overlapping and partly even contradictory. Certain deeply entrenched epistemological beliefs about mathematics, infinity, relations between symbols and things, and other such fundamental categories could explain the choice of one meaning over another in history. These beliefs could be so deeply a part of the mathematicians' thinking that it was impossible or hard for them to conceive of alternative perspectives or to even accept that these beliefs were only beliefs or arbitrary conventions and not some absolute truths, "laws of [the mathematical] nature", so to say. In philosophy, such beliefs about the fundamental categories of thought were called "epistemological obstacles" (Bachelard, 1938). History of mathematical concepts could be written in terms of acts of overcoming such epistemological obstacles, or abolishing the self-imposed artificial constraints. The present day mathematics is a result of a centuries long uphill journey marked by a few turns at which mathematicians would suddenly put into question their practices and habits of thinking and start seeing things from a different standpoint.

In a way, every important mathematical concept or theory was a result of such change of point of view. A new theory often stands on the ruins of some old theory. It can be very difficult to understand the significance of a theoretical edifice without exploring the theories that it has replaced. This may explain some of the difficulties students have understanding courses on theories related to abstract algebraic structures, such as vector spaces or groups. Students are served these powerful syntheses on "empty stomach", as it were. They have never known how it feels to work with, say, infinite-dimensional linear problems trying to generalize the finite-dimensional methods using coordinates and determinants methods. They have not tried to solve functional equations, thought about a possibility to define a distance between functions and engaged with all those questions that have set the ground for the invention of the notion of vector space (Dorier, 2000, p. 54-56). The first historical "examples" of vector spaces were infinite-dimensional normed functional spaces, not the arithmetic R^n spaces that are now considered, in teaching, the paradigmatic examples of a vector space.

We might say that one needs to be a practitioner of a theory in order to become able to think theoretically about it, and thus go beyond only applying the ready made techniques afforded by it.

Thus, understanding mathematical theories requires both practical thinking and theoretical thinking. Practical thinking is the ground against which theoretical thinking acquires its reason of being and without which it loses its epistemological significance. It is in this sense that practical thinking is an *epistemological* obstacle. It is *a cognitive and cultural phenomenon, which stands in the way of certain developments in mathematics, while being, at the same time, an indispensable component of the construction of mathematical knowledge* (Sierpinska, 1990; 1992; 1994).

This is why we chose to formulate our definition of theoretical thinking in terms of its constrast with practical thinking.

A POSTULATED DEFINITION OF THEORETICAL THINKING

Our definition of theoretical thinking has an axiomatic character. It has been inspired by empirical research on students' mathematical thinking but we do not claim that theoretical thinking exists as a separate function of the human organism (like vision, or metabolism). We are thus not trying to *describe* a "higher mental function" of theoretical thinking. We are only postulating a certain theoretical object, which we consider useful as a methodological tool or an analytic instrument. This means, in particular, that our definition cannot be refuted by empirical research. Results of any empirical research on theoretical thinking must be relativized to an assumed definition of theoretical thinking.

We now characterize theoretical thinking as opposed to practical thinking in its aim, object, main concerns and results.

Aim

Theoretical thinking is thinking for the sake of thinking; practical thinking is thinking for the sake of getting things done or making things happen. Theoretical thinking is not aimed at making decisions regarding immediate physical action. Its aim is to understand experience and reflect on the possible outcomes of an action, not to undertake an action.

Object

Practical thinking is thinking about particular "objects" (things, matters, events, people, phenomena). The object of theoretical thinking are systems of concepts.

Main concerns

Meaning of concepts

Theoretical thinking is concerned with *meanings of concepts,* while practical thinking is concerned with the *significance of actions* (see Sierpinska, 1994, p. 24). The main problem of practical thinking is, in general terms, "What should be done to some objects, if one wants some other objects to change in a desired way? Is a given action significant (i.e. important, relevant) with respect to the goal that one wants to reach?" Theoretical thinking asks questions about the assumed meanings of concepts (i.e. their reference and connotation) and the possible consequences of these meanings for the meanings of other, related concepts. Are the concepts well chosen? What is their meaning based on? Are there any implicit assumptions that we have not been aware of? Do the defining conditions lead to a desired meaning?

Conceptual connections

Practical thinking is concerned with factual rather than conceptual connections, e.g., contingency in time and space, analogy between observed circumstances across time, focus on particular examples, focus on personal experience. Theoretical thinking aims at distancing itself from particular events, and particular personal experience, and tries to focus on establishing and studying relations between concepts as they are characterized within a system of other concepts.

Epistemological validity and its hypothetical character

Practical thinking is concerned with *factual validity*: the proof of a plan of action is in the results of the action and in the agreement of the assumptions of the plan with experience, not in the internal coherence among the assumptions, the steps and the expected outcomes. But conceptual coherence and internal consistency of systems of symbolic representations (*epistemological validity*) would be exactly the concern of theoretical thinking.

Aware of its distance to experience, theoretical thinking makes no claims to stating "the truth" about experience. Theoretical thinking produces "propositions" which are conditional or *hypothetical statements.* It is important, for theoretical thinking, to make the assumptions of these statements as explicit as possible. Moreover, theoretical thinking is concerned not only with what might appear as plausible or realistic, but also with what is *hypothetically possible*: thus it tends to analyze all logically possible cases or consequences of an assumption, even if they are practically unlikely.

Methodology

Practical thinking is concerned with the availability of alternative courses of action, if a chosen one does not work. In a way, practical thinking operates always on a single level of its relation with its aims and objects: the level of action whose purposes are external to thinking itself. Theoretical thinking, on the other hand, operates on two levels. It reasons about concepts and it reasons about this reasoning. Theoretical thinking aims at an explicit formulation of its "methodology". In particular, theoretical thinking is concerned with the symbolic notations and forms of graphical representation, and with the rules and principles of reasoning and validation that it uses. It wants to have notations that could be applied for expressing ideas and relations in many areas, rather than use only some ad hoc symbols, different in solving each particular problem. In defining meanings, theoretical thinking is concerned not only with clarity and non-ambiguity, but also with the issues of consistency and independence of the defining conditions.

Results

Results of practical thinking are changes in the objects of this thinking (construction of a new thing, change of a course of events, change in the behavior of people, etc.). Results of theoretical thinking are theories and specialized notations.

In the next section we elaborate on the model in more detail and propose some justification of our claim that theoretical thinking is relevant for the learning of linear algebra.

Theoretical thinking is thinking for the sake of thinking

> [O]f the sciences, the one pursued for its own sake and for the sake of understanding is wisdom to a higher degree than the one pursued for the sake of results from it.
>
> (Aristotle, Metaphysics, Book A, 982a, in Apostle, 1966, p. 14)

In more recent times and closer to educational concerns, the classical Aristotelian distinction between thinking for its own sake and thinking as *poiesis* can be found in Dewey (1933). Dewey defined theoretical thinking as self-serving, purposeful thinking. Theoretical thinking is not concerned with the immediate practicality of life (ibid. p. 222), but it is not the same as daydreaming, because the latter has no purpose. Theoretical thinking is purposeful thinking, and its purpose is not to obtain some "end, good or value beyond itself" but to "facilitate further knowledge, inquiry and speculation" (ibid. p. 222-223).

We also assume that theoretical thinking is voluntary; the thinker is aware of thinking his or her own thoughts, not thoughts imposed by an authority[2]. In particular, a "voluntary" learner of a scientific concept considers him or herself, to a certain extent, an "owner" or a participant in the construction of the meaning of the concept. He or she feels entitled to apply it freely and to change it, adapt it to new problems. This implies the belief that concepts are human creations. This is what distinguishes theoretical knowledge from myth (Steinbring, 1991), religious dogma or the sanctity of a "tradition". In mythical thinking knowledge is justified by the authority of nature (knowledge is in the things themselves, not in our interpretation of the things). In religious dogma, the source of knowledge is in God's revelations. In either case, knowledge needs no justification; it can only be accepted, not questioned or changed. This knowledge can be revered, but not owned by the thinker. Theoretical knowledge is exactly the opposite of these kinds of knowledge: it is regarded as man-made and, therefore, fallible but controllable, requiring constant justification and verification. It is live knowledge.

Let us label the "thinking for the sake of thinking" feature by "reflective thinking".

Linear algebra would not reach its axiomatic form if its inventors were concerned only with the techniques of solving systems of equations and simplifying complicated algebraic

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[2] One inspirational source of this assumption is Vygotsky's postulate of the voluntary character of scientific concepts (1987, p. 169).

expressions such as quadratic forms. Vector space theory is a result of reflection on the existing techniques in the aim of their generalization within the frame of a unified theory. The theory was "self-serving" mathematically: it was not aimed at solving new mathematical problems. As Dorier was saying,

> The [axiomatization of linear algebra] did not, in itself, allow mathematicians to solve new problems; rather, it gave them a more universal approach and language to be used in a variety of contexts (functional analysis, quadratic forms, arithmetic, geometry, etc.). The axiomatic approach was not an absolute necessity, except for problems in non-denumerable infinite dimension, but it became a universal way of thinking and organizing linear algebra. Therefore, the success of axiomatization did not come from the possibility of reaching a solution to unsolved mathematical problems, but from its power of generalization and unification and, consequently, of simplification in the search for methods for solving problems in mathematics.

(Dorier and Sierpinska, 2001, p. 3[3])

The inventors of vector space theory had their practical motivations for this invention and they could see the advantages of the shortcuts afforded by the new structural approach to linear transformations. But it takes a very determined disposition to thinking for the sake of thinking to want to make the effort of learning the products of this invention without prior experience allowing to appreciate their conceptual benefits. It is difficult to see the benefits of vector space theory if all the experience one has is solving small systems of linear equations for which even the matrix representation is superfluous.

Theoretical thinking is thinking about systems of concepts

In introducing his notion of "scientific concepts", Vygotsky (1987) was stressing the assumption that concepts are always part of a system of concepts.

> The key difference in the psychological nature of [scientific and everyday] concepts is a function of the presence or absence of a system. Concepts stand in a different relationship to the object when they exist outside a system than when they enter one.É Outside a system, the only possible connections between concepts are those that exist between the objects themselves, that is, empirical connections. This is the source of the dominance of the logic of action and of syncretic connections of impressions in early childhood. Within a system, relationships between concepts begin to emerge. These relationships mediate the concept’s relationship to the object through its relationship to other concepts.

(Vygotsky, 1987, pp. 234-235).

[3] For a comprehensive account of the historical genesis of the vector space theory, see (Dorier, 2000, Part I).

Theoretical thinking is based on and produces systems of concepts, whose meaning is defined by reference to the meanings of other concepts. In other words, theoretical thinking produces theories.

In theoretical thinking the meaning of a concept is based on its relations with other concepts

Reference "to the meaning of another concept" cannot satisfy the practical mind, which seeks "material" reference at all costs, and attends to anything that may play the role of its substitute: a graphical object such as a line, or an arrow; a symbolic form as a "template" to fill in, a statement as a paradigmatic example of linguistic usage; a catchy phrase; a metaphor.

Students of linear algebra view linear equations and n-tuples as something concrete, as palpable and manipulable linguistic forms. But the "substance" of these familiar objects is illusory: equations represent relations between variables, and the essential thing in n-tuples is not the individual values of the entries but the relations between them.

There have been many attempts in mathematics teaching to introduce abstract mathematical concepts based on the so-called "hands-on experience", thus inadvertently suggesting to the students that their meaning could be founded on empirical relations. Steinbring (1991, 1993) has pointed to the ineffectiveness of these didactic approaches. He studied cases where teachers were suggesting to the students that the mathematical notion of probability can be revealed through the outcomes of natural aleatory experiences (1991), or that the method of derivatives for calculating the dimensions of a box with a maximal volume is "naturally" already there in the measurement of the sizes of the boxes (1993). But the students remained baffled, because their spontaneous ways of thinking were often at odds with the mathematical structure imposed by the teacher (even if these spontaneous ways of thinking made mathematical sense). The students could not "reason out" the connection and thus the relation between the mathematical formula or definition and the physical activity had the flavor of a magical trick for them. They had no control over it.

The epistemological mistake of trying to "derive" abstract mathematical concepts from some more "concrete" experience was also made in the teaching of linear algebra. The "concrete" experience was taking place in the so-called "geometric" or "numerical" approaches, sometimes using technology. The "abstract mathematical concepts" were, for example, the notions of vector, vector space or linear transformation (in its structural form as a linear combinations preserving mapping between vector spaces). As could be expected, the results of these teaching projects were not overly successful in terms of students' understanding of the theory (Sierpinska, Dreyfus and Hillel, 1999; Harel, 2000; Sierpinska, 2000; Dorier & Sierpinska, 2001). Students lived the "concrete experiences" (of manipulating arrows or arrays of numbers) simply as concrete experiences, and not as representations of the general definitions. They were trying to reason about the concepts in terms of some contingent properties of the graphical or numerical representations, irrelevant from the point of view of the concepts, and were falling into contradictions.

In theoretical thinking the meaning of a concept is stabilized by means of definitions using specialized terminology and notation

For Vygotsky, an important consequence of the systemic character of scientific concepts was their sensitivity to contradictions (Vygotsky, 1987, p. 234; see also Sierpinska, 2000). Actually, the very concept of contradiction does not make sense outside of a theoretical system of concepts. Contradiction is a type of logical relationship between propositions; there can be no contradiction between things or events occurring in space and time. Things and events cannot be contradictory because their meanings change over time and contexts.

In practical thinking judgment is applied to people's behavior and utterances in social situations, not to statements. Two people may have very different definitions of a situation in which they find themselves (for example, a teacher and a student in the classroom). But, depending on their "predilection to agreement" they may reach some "contract of agreement" through which a "social meaning" can be developed. As mentioned in the presentation of our theoretical framework, this social meaning has little to do with "epistemological truth or falsity" and a lot to do with a common sense of what is "real" and what is "illusory", "makes sense" or "is preposterous", etc. Contradiction in practical thinking emerges for the participants in a situation as a "disagreement of standpoints", a "breaking of the contract", an act of trespassing of a norm. Thus contradiction in practical thinking is an emergent and relative phenomenon, i.e. it depends on the people involved in a situation, their definitions of the situation which can influence their interpretation of past events and their expectations about future events. (McHugh, 1968, pp. 30-31, 35).

From the point of view of theoretical thinking, the possibility to speak about contradiction presupposes a certain *stability of meanings* in the frame of reasoning, and reasoning is the very essence of systemic thinking. Focused on relations between concepts, having no other access to reality but through discursive representations, theoretical thinking must rely on reasoning to achieve knowledge. It cannot validate a statement just by saying that it agrees with what everyone can see. But the validity of a chain of reasonings depends on the stability of meanings between one link and another. This stability is achieved through *definitions*.

For some authors in mathematics education, reference to definitions served as a demarcation line between "advanced mathematical thinking" and "elementary mathematical thinking":

> The move from elementary to advanced mathematical thinking involves a significant transition: that from describing to defining, from convincing to proving in a logical manner based on those definitionsÉIt is the transition from the coherence of elementary mathematics to the consequence of advanced mathematics, based on abstract entities which the individual must construct through deductions from formal definitions.
>
> (Tall, 1991, p. 20)

In "advanced mathematical thinking", definitions create "formal concepts", whose nature is different from that of "procepts", amalgams of processes, concepts and symbols, which dominate in elementary mathematics (Gray, Pinto, Pitta and Tall, 1999).

"Formal concepts" have much in common with Vygotsky’s notion of "scientific concept" and the notion of "formal category" in the sense of Bruner, Goodnow and Austin (1960). Bruner et al. distinguished formal categories from "affective" and "functional" categories, which are widespread in everyday communication. Formal categories are determined by explicit defining properties, while the meanings of "affective" and "functional" categories remain implicit and context-bound.

Linear algebra students, who complain about their difficulties with the so-called "proof problems", often say that they "don’t even know how to start". When advised to think about the meanings of the concepts involved in the statement, they rarely think about the definitional conditions that these concepts satisfy. They think about some examples and this, of course, cannot lead them to producing a proof of a general statement (although it can be helpful in disproving a general statement).

The conditions of a definition are sometimes the result of a process of categorization in a domain of existing concepts. We assume that, in theoretical thinking, this categorization is "systemic", i.e. it has a "key", or a well-defined feature, which serves as a basis for separating concepts into disjoint classes. Everyday concepts can rarely allow for such systemic categorizations, because everyday concepts are more often *complexes* in Vygotsky’s sense (Vygotsky, 1987, p. 136) than scientific concepts. Classifications based on complexive thinking do not take care of separating a whole into a union of disjoint classes; elements are grouped together based on contextual and functional associations, and the reasons for putting an item into a group may vary between one group and another (examples of students' non-systemic categorizations can be found, e.g. in Sierpinska, 1994, p. 146).

Concerned with theoretical foundations, vector space based linear algebra courses do not stop to look at applications, but go on with the development of the theory, introducing ever new definitions and theorems. Once a series of concepts has been introduced, a categorization of these concepts follows, thereby allowing the theory to develop further with a more manageable set of fewer, but more general concepts. Classification of quadratic forms, categorization of matrices into diagonalizable and non-diagonalizable and further into block-diagonalizable according to certain canonical forms, are but a few examples of this activity of categorization. The effort of constructing these categorizations is certainly not comparable to the effort of learning the resulting concepts. Still, the learners would be quite helpless in front of the task of understanding the concepts if they were not able to engage in systemic categorization themselves. The learner needs this ability for "sorting out" the concepts, making sense of how they are related to each other. Of course, the spontaneous students' categorizations may be based on different criteria than those chosen in the lecture. Students may have difficulty understanding why some particular criteria were chosen, and the categories proposed by the lecturer may not make sense to them. Some students make up for the lack of conceptual categorizations by connecting concepts on the basis of a variety of situational associations (e.g. "this is something we did in week 2, and this other thing was given for homework in week 4"), or associations with memorized phrases or labels (e.g., names of theorems, titles of chapters), but such categorizations are not very useful in understanding and writing proofs.

Theoretical thinking in mathematics entails acceptance and understanding of axiomatic definitions

We have assumed that, in theoretical thinking, meaning is found in definitions of concepts. What kind of definitions do we have in mind?

Logicians have identified many kinds of definitions: ostensive, contextual, stipulative, operational, analytic, nominal, lexical, axiomatic, by abstraction, recursive, classical, normal, and other (*Mala Encyklopedia Logiki*, 1988). For example, in the cited Encyclopedia of Logic, a "normal definition" is defined as follows: a statement D is a normal definition of an expression E in a language L if D has the form of an equality or equivalence, which makes it possible to translate every phrase in L containing E into a phrase in L not containing E". A "normal definition" is "nominal" in the sense that it defines one linguistic expression by another linguistic expression and, in principle, may give no hints as to the possible referents of either expression.

Contrary to a normal definition, the so-called "classical" or "real" definition intends to point to the denotation of an expression: the objects, to which this expression can be applied. The classical definition is of the form: "A is B and C", where C is a subclass of B. B is then called the kind to which all elements of A belong and C specifies the characteristics that distinguish the elements of A among the elements of B. For example, in the definition "a natural number is prime if and only if it has exactly two divisors", A = prime numbers, B = natural numbers, and C = natural numbers which have exactly two divisors. One could write the definition in a more explicitly "classical form": a prime number is a natural number which has exactly two divisors. Such definitions have been called "classical" because they have much in common with what Aristotle understood by a definition in his "Topics"[4]. But these definitions have also been called "real" because they were meant to refer to some existing reality, whether physical or conceptual.

The distinction between the nominal and the real definitions might seem quite clear from the above, but one immediately starts having doubts when applying this distinction to mathematics. The example we gave of a real definition of prime numbers - "a natural number is prime iff it has exactly two divisors" - could be easily turned into a nominal definition: "prime number" = "natural number which has exactly two divisors". This is an equivalence of two linguistic expressions; the definition states that they are synonyms and, henceforth, "prime number" can replace the phrase "natural number which has exactly two divisors".

Whether one understands the definition of prime numbers as nominal (equivalence of linguistic expressions) or real (giving information about the objects, to which the term "prime numbers" can be applied) is a matter of a philosophical choice concerning the ontology of mathematical objects: formalistic or realistic. We assume that theoretical thinking leans more towards formalistic than realistic ontology. But we also assume that *theoretical thinking is not*

[4] Aristotle's aim in "Topics" was not so much to propose a "theory of definitions" but to distinguish between definitions, properties, kinds and incidental features of things (*Topics*, Book I, 101-102). His goal was to educate a critical listener of the "proofs of sophists" who would not be fooled by their rhetoric. One of the rhetorical tricks is to characterize a thing by its property or even an incidental

concerned with the questions of ontology, while ontology is most important in practical thinking. The practical thinker is not interested in objects whose existence is only formal.

From the point of view of theoretical thinking, the question is not whether or not the defined object enjoys a real or a formal existence (i.e. whether the definition is real or nominal), but *whether a definition offers a mere terminological shortcut or indeed contributes to the growth of the theory in some essential way* (Grzegorczyk, 1981, p. 203). For example, the definition of linearly independent vectors in R^n is reducible to the concept, "a homogeneous system of equations with only the trivial solution". Thus it could be regarded, from the point of view of theoretical thinking, as a mere terminological shortcut. But the definition of a linearly independent set of vectors in an arbitrary vector space is not reducible to a previously introduced concept. Therefore, it obtains the status of a theory-building concept. Another example: The definition of R^n can be regarded as a mere "summary" of the known properties of operations on n-tuples for n= 2, 3, and, perhaps 4 or 5. In contrast, the definition of the general vector space extends the domain of linear algebra in a substantial way: it introduces new primitive terms (e.g., "vector", which may sound familiar but is, in fact, a new primitive term that refers to a member of a "vector space", i.e., any collection of objects in which certain operations have been defined, satisfying certain properties) and opens the gate for the construction of new, unanticipated, objects. Collections of functions with the operations of addition and multiplication by a scalar can be regarded as vector spaces, and the concept of vector space can be applied in the theory of differential equations.

Thus, from the point of view of theoretical thinking, particularly in mathematics, the most important distinction is not between real and nominal definitions but between terminological-shortcut definitions and axiomatic definitions. Acceptance and understanding of axiomatic definitions would enable linear algebra students not only to verify that a given object is, say, a vector space or a linear mapping, but also construct / invent new examples of vector spaces or linear mappings, and solve problems where the unknown would be such an axiomatically defined object. Exercises in verification whether a given object satisfies a definition can be successfully done by students who do not understand the "non-real" character of these definitions. Success in construction problems is much more difficult to obtain (see Sierpinska, Dreyfus, Hillel, 1999).

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feature and pretend that one is pointing to the very essence of the thing as if one were giving its definition.

In our experimentation of teaching linear algebra in the Cabri environment (Sierpinska, 2000) many students were quite confused in their understanding and use of the definition of linear transformation. Students were shown examples and non-examples of linear transformations and were encouraged to verify, which of these examples satisfied the definition. It would be logically acceptable to conclude, from these examples, that if a transformation is, let's say, a combination of a rotation and a dilation, then it is a linear transformation. But the students unexpectedly (for us) concluded that linear transformations are exactly those transformations that are combinations of rotations and dilations. This notion allowed them to solve some problems involving concrete vectors and transformations. But students felt completely clueless when confronted with the problem of defining the value of a linear transformation on an arbitrary vector, given its values on a basis. For example, given the values w_1 and w_2 of a plane linear transformation T on non-collinear vectors v_1 and v_2, respectively, the students would easily find the values of the transformation on some concrete linear combinations of the vectors v_1 and v_2 (e.g. $1.5v_1 - 0.8v_2$) but they were at a loss when asked to define the value of T on an arbitrary vector v in the plane. Rather than representing v as a linear combination $a_1v_1 + a_2v_2$ and then constructing T(v) as $a_1w_1 + a_2w_2$, most students were trying to find out, from the relations between the vectors v_i and w_i, "what is T": is it a rotation, a dilation, a combination of the two? These students appeared to perceive the defining condition of linear transformations as only a property of a transformation, which does not, in fact, "define" it, properly speaking. Tendency to think in terms of prototypical examples, rather than in terms of a general definition, about linear transformations, made it very difficult for the students to understand the notion of matrix representation of a linear transformation, which requires seeing linear transformations on finite-dimensional spaces as *fully determined* by their values on a basis.

Perhaps the students' understanding of linear transformations as "dilations, rotations, shears, and their combinations" could be explained by their lack of distinction between conditional and biconditional statements. But another reason could be the students' reluctance to even accept definitions that were not describing the reference of a word, or that did not just give a general name to a set of already existing (and well known) objects. They had the practical person's attitude to the question of ontology of mathematical objects, and had a tendency to interpret an axiomatic definition as describing some already existing reality rather than as *postulating* a new object, of which they knew but a few examples.

But axiomatic definitions abound in linear algebra courses focused on the construction of the theory of general vector spaces, even if the field of scalars is not arbitrary but is assumed to be the field of real numbers. This focus on careful definitions, subtle logical distinctions, nit-picky study of pathological cases which do not satisfy a definition and thus justify one or another of its conditions, etc., is hard to understand by a practically thinking student, concerned mainly with ontology, i.e. the existence of sensible, meaningful examples.

Theoretical thinking is concerned with internal coherence of conceptual systems

Theoretical thinking treats definitions as conventional, arbitrary, the only restriction being that the defining conditions be non-contradictory, and that the definition does not contradict other statements in the system. Contradiction would mean that the definition is "empty" or that the object it defines does not "exist".

In a sense, theoretical thinking is "narcissistic": concerned with itself, its own internal consistency. It interprets problems of existence as problems of *non-contradiction*, and problems of truth — as problems of *validation*. In mathematics, a statement is considered true if it is provable within a system of concepts. The meaning of "valid" is also an object of reflection in theoretical thinking, which leads to the development of "methodologies" (theories of methods of validation, specific to a domain of knowledge).

On the other hand, practical thinking is oriented to the outside: it looks for external evidence in its claims about existence and validity: concrete examples, agreement of expectations of the results of an action with the actual results, etc. Practical validity is about being "right", "fair (or good) enough", not about being "true" within the frame of a conceptual system. In a social situation, a statement may be considered "right" if an agreement has been achieved among the participants, that it is acceptable with respect to some spoken or unspoken norms, under the circumstances. (McHugh, 1968, p. 35; see also Voigt, 1995).

Concern with coherence may require restructuration of the conceptual system whenever a new concept is added

Concern with internal coherence of conceptual systems implies that theories do not grow by simple addition of new concepts and/or propositions. Of course, if the new concept only brings together an existing set of relations and gives them a name, we might speak of a "simple addition". But in cases where the new concept reveals unnoticed relations and possibilities, the whole conceptual system must be changed, for the sake of its consistency, its generality and all those features that make a system something more than a collection of interesting ideas and metaphors. For example, the axiomatization of the notion of vector space (in 1920's) entailed a redefinition of most concepts developed thus far in linear algebra and important shifts in the "relevance ranking" of concepts took place. Matrices and determinants, once the main objects of linear algebra, were relegated to the rank of technical tools, useful only under very strong assumptions. Linear transformations took over the position of the main object, but, in the new theory, they were no longer just "linear substitutions of variables", equivalent, in fact, to systems of equations or matrix operators. The new theory populated linear algebra with infinite dimensional vector spaces and their transformations, and thus with transformations which could not be reduced to a multiplication by a matrix.

Theoretical thinking is hypothetical

Another implication of the way theoretical thinking conceives of the notions of meaning and truth is what we have called its hypothetical character.

Thinking within conceptual systems can only produce conditional truths. Theoretical thinking is assumed to be concerned with making explicit the assumptions of these conditional truths. It is especially distrustful of the obvious, the immediately certain, which is the basis of all practical thinking. Practical thinking sees universality in the concrete detail; it extrapolates generality from a single event and tends to build a whole story from a few bits of information. By contrast, theoretical thinking, which is sensitive to the linguistic form and expression, analyzes information for what it is worth and tries to avoid adding to it what has not been said.

But by seeking to uncover its foundations and implicit assumptions, theoretical thinking makes them liable to change and speculation. "What if we assumed something else?" is its main question. Rejecting an assumption just because it "sounds unrealistic" is not in the style of theoretical thinking. Bound by its systemic character, theoretical thinking wants to exhaust all possibilities and consider all cases. As Bachelard (1934), was saying, the driving force of the theorizing thought is not the question "why?" but the question "why not?" which leads to an ample surplus of generalizations and alternative theories (p. 10-11). Some of these generalizations are then rejected or forgotten, by the interference of the practical side of the minds of scientists (the famous "Okham's razor"), as uninteresting, or irrelevant.

From the practical point of view, many if not most of the questions discussed in a linear algebra course are futile (or "frivolous", as one of the interviewed students put it). To take an example from the beginning of a linear algebra course, a practically thinking student might ask, Why all this fuss about the non-commutativity of multiplication in matrices? What is the point of this? You multiply two matrices, you get a result, and you go on to doing something with this result; you don’t sit there thinking what would happen if you multiplied the matrices the other way round, because multiplying them the other way round wouldn’t make sense from the point of view of the particular data you are dealing with. But, in linear algebra, questions of the type, "what would happen ifÉ form the backbone of the whole theory.

Perhaps the following image explains best what we mean by the "hypothetical" attitude of theoretical thinkers: when a practically thinking person (P) and a theoretically thinking person (T) play a game, P rejoices when she wins and is sad when she loses, and wants to play again. T neither rejoices nor gets sad, but starts thinking about the conditions of winning the game, given the rules of the game and different strategies. Is there a winning strategy or not? Could the rules of the game be changed so that there is a winning strategy?

Theoretical thinking takes an analytical approach to signs

Bruner et al. (1960) stressed that the explicit character of formal categories requires the development of specialized representational systems (terminologies, symbolic notations).

> Oftentimes the careful specification of defining properties [of a formal category] even requires the construction of special ’artificial’ languages to indicate that common-sense functional categories are not being used. The concept ’force’ in physics and word standing for the functional class of events called ’force’ in common sense do not have the same kind of definition.É[F]ormal categories and formal category systems appear to develop concurrently with methods for representing and manipulating them symbolically.É [S]ymbolic representations of formal categories and formal category systems are eventually developed without reference to the classes of environmental events that the formal categories ’stand for’. Geometry provides a case in point, and while it is true that its original development was contingent upon the utilitarian triangulation systems used for re-dividing plots after floods in the Nile Valley, it is now the case that geometers proceed without regard for the fit of their formal categories to specific empirical problems.

(Bruner et al., ibid, p. 5-6)

If meanings of signs are derived from their relations with other signs then this meaning is based on certain explicit conventions and not on, say, some "natural" resemblance or contiguity between signs and their intended objects. But one convention can always be replaced by another one. This creates a distance between signs and objects in theoretical thinking: the relation is *indirect, mediated* by a language. This is what we mean by "*analytic* *approach to signs*", to which we shall sometimes refer as "analytic thinking", for the sake of brevity. It is a very different cognitive activity altogether to imagine a boundless flat surface than to think about capturing this image in a definition such as "a plane in a 3-dimensional Cartesian space is the set of all ordered triples (x, y, z) of real numbers for which there exist real numbers a, b, c and d such that ax + by + cz + d = 0", realizing, at the same time, that other possible representations are possible (e.g., in the language of vectors).

We emphasize the use of the indefinite article "a" in saying that, in theoretical thinking, the relation between a sign and its intended object is mediated by *a* language. We wish to stress that theoretical thinking does not take language for granted. Language is not, as in practical thinking, only a tool for making things happen (although it is that, too, in theoretical thinking, especially when the language is a notation such as in algebra or in predicate calculus). It is treated as an object of study and development in its own right. To borrow a phrase from Marody (1987, p. 186), we could say that theoretical thinking is thinking "in a language" while practical thinking is thinking "through language". In theoretical thinking, "language contains a meta-linguistic layer which allows one to speak and think about a given form of language as one of many possible forms of language" (ibid.). This opens the way to the creation of specialized forms of discourse, in which new words are invented, and old words are used in new meanings.

Sociologists of language have been linking the historical development of what we have called "analytic approach to signs" with the expansion of the written language (Marody, ibid. p. 51). Also Vygotsky insisted on the role of written language in the genetic development of scientific concepts. For him, written language was "the algebra of speech".

> Even the most minimal level of development of written speech requires a high degree of abstraction.É Written speech is speech in thought, in representations.É [W]ritten speech differs from oral speech in the same way that abstract thinking differs from graphic thinking.É Written speech is more abstract than oral speech in other respects as well. It is speech without an interlocutor.ÉSpeech that lacks real sound (speech that is only represented or thought and therefore requires the symbolization of sound —a second order symbolization) will be more difficult than oral speech to the same degree that algebra is more difficult for the child than arithmetic. Written speech is the algebra of speech. É [O]ral speech is regulated by the dynamics of the situation.É With written speech, on the other hand, we are forced to create the situation or — more accurately — to represent it in thought. The use of written speech presupposes a fundamentally different relationship to the situation, one that is freer, more independent, more voluntary.

(Vygotsky, 1987, ibid. p. 202-203)

By creating written, and, therefore, static and decontextualized representation of speech, language could become an object of analysis. It became possible to identify and codify discursive patterns and, eventually, to create grammar and formal logic.

Being sensitive to linguistic patterns in mathematics implies identifying the special role that the quantifying expressions such as "for any", "for some", "there exist" play in mathematical statements. It implies noticing that mathematical statements are always conditional, even those that are not formulated explicitly in the "if Éthen" form. It also means having a sense that the converse of a conditional statement is not always true and that, to show that a conditional statement is false it is enough to find an object which satisfies the premise but not the conclusion[5].

At the level of statements, "grammatical sensitivity" implies noticing that there are various types of statements, labeled as "definitions", "examples", "axioms", "propositions", "theorems", or "proofs", which differ with respect to their format, logical structure, the way they are justified and the roles they play in the mathematical discourse. In the history of mathematics, this analytic sensitivity to mathematical language has led to the development of a whole new domain of knowledge — metamathematics — with its *theory of proof* and *theory of definability*. Theoretical reflection on the methodology of mathematics and theoretical knowledge in general (thus a "second order" theoretical thinking) has produced many a subtle conceptual distinction.

We shall divide the feature of analytic thinking into two main aspects: "linguistic sensitivity" and "meta-linguistic sensitivity". A theoretically thinking student of mathematics will be assumed to have a certain "linguistic sensitivity" in mathematics, i.e. sensitivity to the syntax of the formal symbolic notations, and the usage of mathematical specialized terminology in less formal sentences. He or she will also be assumed to have some "meta-linguistic sensitivity" to the arbitrary, conventional relation between mathematical signs and their referents ("symbolic distance"), to logic or rules of inferring new statements from given ones, and to parts of "mathematical speech", i.e. axioms, definitions, theorems, proofs, explanations, examples, etc.

A priori, analytic approach to signs is extremely relevant for the learning of linear algebra, because linear algebra could be seen, in fact, as all about languages and expressing

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[5] Some students in advanced mathematics courses seem to have figured out these "laws of logic" by just observing their teachers' mathematical behavior and studying proofs in textbooks.

the same thing in many different ways: it was born from the need to express certain complicated symbolic forms in a simpler way. Besides using the formal language of logic and set theory, linear algebra uses the "geometric", the "algebraic" and the "abstract" languages (Hillel, 2000), thus creating several "registers" of representation, such as the "graphical" (arrows, drawings of lines and planes), the "tabular" (matrices) and the "symbolic" (axiomatic theory of vector spaces) registers (Pavlopoulou, 1993). Subspaces may be represented using Cartesian or parametric descriptions. This makes linear algebra indeed an "explosive compound" of languages, which are in constant interaction and require much cognitive flexibility (Alves Dias & Artigue, 1995). This flexibility cannot be achieved without distancing oneself from the linear algebra languages and seeing them as objects of study in their own right.

Linear algebra requires a great awareness of the arbitrary, conventional, and temporary character of the meaning of signs used therein. While, in high school algebra, the meaning of certain letter symbols could be fairly stable, like, for example, "x" always representing the independent variable, "y" — the dependent variable, "a", "b" and "c" — constants or parameters, in the linear algebra course, much of this kind of stability is lost. Students have to be prepared that the notation used by the instructor may be different from that used in the test questions, written by a different professor. And the instructor may also use different notations for the same object depending on which is more convenient in a given situation.

For example, vectors in the R^n spaces (n=1, 2, …) are written as (a_1, a_2), (x, y, z), $(x_1, \ldots, x_n)$, $(v_1, \ldots, v_n)$, $[x_1, \ldots, x_n]$, as column vectors, etc. They are said to have "components", "coordinates" or "entries". Square and round brackets are used interchangeably for vectors, but the curly brackets, "{", in the context of vectors denote a totally different object: a set of vectors. A student who does not pay sufficient attention to the way things are written may fail to distinguish between the symbol $(v_1, \ldots, v_n)$ (a vector with "components" or "coordinates" $v_1, \ldots, v_n$) and the symbol $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$, representing a set of vectors.

The differences in terminology and notation are not just a matter of names and shapes of signs. Different terminology or notation often implies a different concept of a mathematical object. For example, one might say that there is one mathematical *object* "vector" in R^n but there exist several different *concepts* of it: for example, an ordered set of n numbers, which has "components"; or a "coordinate vector", if one looks at vectors as combinations of vectors of some basis. One can also look at elements of R^n as one-column matrices, which have "entries". It is this *coupling of linguistic sensitivity with conceptual understanding* of signs that makes analytic thinking so important in linear algebra.

We assumed that analytic thinking entails sensitivity to the logic of the mathematical discourse. There have been hopes that this sensitivity can be developed by explicit teaching of logic courses. But these hopes turned out to be unjustified (Dorier and Sierpinska, 2001, p. 271; see also Selden and Selden, 1996). Knowledge of logical laws alone is not sufficient to make conceptual connections and to verify their validity, because logic is restricted to form and conceptual connections are not. In formal logic, the difference between the premise and the conclusion is in their formal position before or after the implication sign. In the context of concrete conceptual domains, the conclusion has also a meaning and a significance (i.e., a value or relevance as a piece of knowledge). Also, logic focuses on proving tautologies. But in content domains, tautologies are not interesting. In linear algebra, the interesting part is to be able to talk about one thing in a variety of ways, whose meanings overlap only partially. It is enough to think of the notion of linear dependence and try to speak about it in the contexts of non-trivial solutions to homogeneous systems of linear equations, general vector spaces and bases, singular matrices, determinants, existence of eigenvalues.

There is a long way from knowing truth tables for implication and knowing how to negate an implication to understanding the notion of mathematical necessity. In logic courses students learn to say that if p implies q, then q is a *necessary* condition for p. But they often fail to see that, in the definition of linear independence, the condition that

$$a_1 v_1 + \ldots + a_n v_n = 0 \text{ implies } a_1 = \ldots = a_n = 0$$

means that the trivial solution is the *only possible* one. The notion of necessary condition acquires meaning in contexts, where missing it is likely to lead to erroneous conclusions.

SUMMARY OF OUR ARGUMENTS FOR THE NECESSITY OF THEORETICAL THINKING IN UNDERSTANDING LINEAR ALGEBRA

The undergraduate learner of linear algebra must be even more theoretically inclined than the inventors of the theory.

Linear algebra, as a theory of vector spaces and linear transformations, cannot be understood by the undergraduate student in an act of "overcoming an obstacle", against some previously conceived knowledge, by realizing that it is insufficient or too awkward for dealing with certain new interesting problems; it cannot be "constructed" in the same way as it was invented. This knowledge and these problems are not accessible to the undergraduate student. Thus the student must engage with the concepts of the theory without apparent motivation drawn from his or her own mathematical experience, studying them for the sake of these concepts alone, seeking to uncover their intrinsic significance within the theory itself.

Meaning of concepts must be sought in their relations with other concepts

The student who seeks the meaning of concepts such as vector, vector space or linear transformation in some more familiar realm, e.g. graphical (drawings of arrows, lines and planes) or numerical (operations on matrices) is quickly deceived by his or her misconceptions. Unlike the historical obstacles of the awkward generalizations of the theory of determinants to infinite cases, these experiences cannot become the (practical) knowledge, against which one could build a vector space theory. The novice in linear algebra has no other choice but to construct meanings on the basis of the existing definitions and theorems. At the beginning learning may feel like trying to find one’s way around a dark and unfamiliar attic. But, eventually, the student starts to know enough about the objects of the theory to have some control over the directions of his or her movements: this happens when the student formulates his or her first theoretical problem, questioning a conclusion, asking why a given assumption has been made, or why a lengthy proof cannot be made simpler[6].

The learner must engage in proving activity and therefore use systemic approaches to meanings and validity

Linear algebra courses focused on vector space theory are concerned mainly with their own foundations, not with the possible applications, whence the stress on proofs. Proofs as an exercise for students and not just part of the lecturer’s job are a trademark of linear algebra courses, distinguishing them from other undergraduate courses, such as Calculus, Differential Equations, or Probability and Statistics. Disproving a general statement in a "True or false" type of questions can sometimes be managed using a "paradigmatic example" approach to

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[6] Here is an example of such "first" student question: "Why the Cayley-Hamilton theorem (every square matrix A is a root of its characteristic polynomial, det(A — xI)) cannot be proved simply by substituting A for x in A — xI; since A — AI = O, then, of course, the determinant is zero and the proof is done".

meanings, but proving a general property requires making connections between concepts based on definitions and theorems.

The learner has to accept that his or her ontological questions will remain unanswered

Students often ask, "But WHAT is a vector space, after all?" Seeing the axioms as a set of well-known and trivial properties, they are not satisfied with answers such as, "A vector space is any system satisfying these conditions". More satisfactory would be answers such as, "Vector space is a general name for systems such as R^n (the vector space of n-tuples over real numbers), R_{nxm} (n x m matrices over real numbers), P_n (polynomials of degree † n), C[a,b] (continuous functions on the interval [a,b])". The practical thinker finds it preposterous to give a name to objects that do not exist yet, but that could be constructed so as to satisfy the axioms. But the learner of linear algebra must be prepared to construct new vector spaces, perhaps with some additional requirements (like, e.g. an inner product). This freedom of constructing new objects in mathematics is sometimes too overwhelming for the novice, evoking an unpleasant sensation of lack of support, rather than a feeling of liberation from constraints. In reaction to an example of a vector space not belonging to the range of the paradigmatic examples, some students ask, with an expression of distrust, "So you say you made it up, invented it, just like that? But then, is it a real vector space?" As if you needed a brand name on your vector space for it to count as a "real" vector space.

The learner must engage in hypothetical thinking

Engaging with linear algebra theory means asking not only questions such as, "What are the implications of these assumptions?" but also, "What would be the implications of a slightly different set of assumptions?" Suppose we allow the field of scalars to be rational numbers rather than real numbers. Would then the given linear operator still have a Jordan Canonical form?

The learner must become mathematically "multilingual"

Linear algebra is a result of an enormous creativity in the field of mathematical notations and of a search for more effective, concise ways of representing relations between large numbers of variables. The learner must therefore not only have a "talent for languages" as a user of the idiom; he or she must be disposed to think about the structure and form of the languages and the implicit rules governing the specialized discourses.

SUMMARY OF THE POSTULATED FEATURES OF THEORETICAL THINKING

We have analyzed our model of theoretical thinking into a number of features, which we list below using labels such as "reflective", "systemic", "analytic", etc. We shall use these labels in referring to these features in operationalizing the model in the next section for the purposes of empirical research.

TT1 REFLECTIVE: Theoretical thinking is thinking for the sake of thinking.

TT2 SYSTEMIC: Theoretical thinking is thinking about systems of concepts, where the meaning of a concept is established based on its relations with other concepts and not with things or events

- TT21 DEFINITIONAL: The meanings of concepts are stabilized by means of definitions
- TT22 PROVING: Theoretical thinking is concerned with the internal coherence of conceptual systems
- TT23 HYPOTHETICAL: Theoretical thinking is aware of the conditional character of its statements; it seeks to uncover implicit assumptions and study all logically conceivable cases

TT3 ANALYTIC: Theoretical thinking has an analytical approach to signs

- TT31 LINGUISTIC SENSITIVITY
  - TT311 Sensitivity to formal symbolic notations
  - TT312 Sensitivity to specialized terminology
- TT32 META-LINGUISTIC SENSITIVITY
  - TT321 Symbolic distance between sign and object
  - TT322 Sensitivity to the structure and logic of mathematical language

CHAPTER II. INTERVIEWS WITH A GROUP OF HIGH ACHIEVERS

In the winter term of year 2000 we interviewed 14 students who achieved high grades in a first linear algebra course in our university. The course was intended for students specializing in mathematics or actuarial mathematics. All but one of these students were in the actuarial mathematics program. This student intended to return to work in the military sector where she had worked before coming to the university. Twelve of the students were interviewed in February, at the time when ten of them were taking the second linear algebra course, one was doing her internship in a company, and one switched from mathematics to psychology. Two remaining students were interviewed in May, after they had completed both courses.

We were interested in knowing if students who were awarded high grades in their first linear algebra course had a high disposition to theoretical thinking.

The interviews were based on seven questions, not all related to linear algebra concepts, but aimed at probing what we considered, at that time, the main features of theoretical thinking. The model of theoretical thinking that we had at the time of the interviews was only slightly more developed than the one we proposed in (Sierpinska, 2000). By designing, conducting and analyzing the interviews we hoped to refine the model and develop a methodology for evaluating a student's disposition to theoretical thinking based on that refined model.

DESIGN OF THE INTERVIEW QUESTIONS AND METHOD OF CODING STUDENTS' RESPONSES

We wanted to have questions that would provoke a distinctly different behavior in a theoretically thinking person than in a practically thinking person (according to a given model of theoretical thinking). Seven questions were designed, based on our initial model. As we analyzed and re-analyzed the students' responses to these questions, we felt the need to refine some aspects of the model. The model presented in Chapter I is a result of this work of refinement. We realize that the interview questions could be better designed to represent the refined model. The design of a revised set of questions is planned to be undertaken in a future research. In this chapter, we present and analyze the questions we used in our interviews, from the point of view of the refined model.

Students’ responses to the questions were evaluated based on the refined model. Our presentation of the interview questions in this chapter contains a description of students’ behaviors that were considered to represent theoretical thinking, practical thinking or a mixture of the two. The features of the students’ behavior were coded so as to indicate, which feature of theoretical thinking in our model they were assumed to represent. For example, the behavior, "the student does more than just solve a given problem to the satisfaction of the interviewer; he or she poses interesting new questions and conjectures", would be labeled as "researcher’s attitude" and coded as TB1a. The code was meant to signal the assumption that the behavior was a feature of "theoretical behavior" corresponding to the feature TT1 (Reflective thinking) of the model. The behavior, "the student appreciates the intrinsic significance of mathematical concepts" was coded as TB1b, and assumed to be another behavioral symptom of the TT1 feature of theoretical thinking.

We identified 18 "theoretical behavior" (TB) features. We list them below, in the order of the features of theoretical thinking they were assumed to express.

TT1 REFLECTIVE

- TB1a Displaying an investigative ("researcher’s") attitude towards mathematical problems
- TB1b Appreciating intrinsic significance of mathematical concepts
- TB1c Using relational discourse

TT2 SYSTEMIC

- TT21 DEFINITIONAL
  - TB21a Using formal categorization
  - TB21b Referring to definitions in algebraic contexts when deciding upon meanings
  - TB21c Referring to definitions in graphical contexts when deciding upon meanings
- TT22 PROVING
  - TB22a Engaging in proving activity
  - TB22b Refuting a general statement by drawing a contradiction
  - TB22c Engaging in axiomatic reasoning
- TT23 HYPOTHETICAL
  - TB23a Being aware of the conditional character of mathematical statements and engaging in discussions about the possible consequences of adopting different sets of assumptions

TT3 ANALYTIC

TT31 LINGUISTIC SENSITIVITY

TT311 SENSITIVITY TO FORMAL SYMBOLIC NOTATIONS

TB311a Interpreting algebraic expressions in a rigorous way

TT312 SENSITIVITY TO SPECIALIZED TERMINOLOGY

TB312a Being articulate and using correct terminology

TT32 META-LINGUISTIC SENSITIVITY

TT321 SYMBOLIC DISTANCE BETWEEN SIGN AND OBJECT

TB321a Interpreting letters in algebraic expressions as variables

TB321b Interpreting graphs as representing relationships between variables

TT322 SENSITIVITY TO THE STRUCTURE AND LOGIC OF MATHEMATICAL LANGUAGE

TB322a Being aware of the role and meaning of expressions such as "for all", "for some"; having a sense of the implicit universal quantification of variables in conditional statements; negating the universal quantifier by the existential one and vice versa

TB322b Being sensitive to logical connectives; in particular, to implication and its negation in the definitions of linear independence and dependence

TB322c Being sensitive to the form of definitions and distinguishing them from explanations

TB322d Distinguishing between an implication and an equivalence statement

For each question, we identified the TB features that were revealed through this question in all students. We then coded each student's behavior in a given question with respect to a given feature by the vector [1,0] if his or her behavior was consistently theoretical, by [0,1] if the behavior was consistently practical. The vector [1,1] was assigned if the student's behavior was, at times, theoretical, and, at other times, practical, which could happen when the student changed his or her approach in the wake of the interviewer's remark or of a conflict between his or her expectations and the results in a computation. This led to a table whose rows corresponded to the 14 students, and columns - to TB features in each question (the "Question-by-question table", see Appendix I).

On the basis of this table, the "Feature-by-feature" table was obtained (see Appendix II). The columns of this table represented the TB features; the rows represented the 14 students. If a feature was observable in two or more questions, it was counted only once. More precisely, we adopted the following rule of calculation: the vectors were added coordinate-wise so that

+	0	1
0	0	1
1	1	1

The students were coded O1, O2, O3, O4, V1, V2, V3, V4, S1, S2, S3, S4, N1, and N2, using the initials of the last names of the interviewers (Asuman *O*kta , Alexe *V*olkov, Anna *S*ierpinska, and Alfred *N*nadozie).

ANALYSIS OF THE INTERVIEW QUESTIONS AND STUDENTS' RESPONSES

Question 1. "Classification"

The first question asked the students to classify a set of 5 algebraic expressions

$2x + 3 = 7$ $\quad$ $y = x^2 - 4x + 1$ $\quad$ $A = ah/2$ $\quad$ $2x - x^3 + 78$ $\quad$ $x - 13 = x$

into at least two groups according to their own criteria. The question was worded as follows: *"Here is a pile of expressions, each written on a separate card. You are asked to separate these expressions into at least two groups, according to your own criteria*." The students were asked to explain what were their criteria in classifying the items.

Analysis of Question 1. This was a classical type of question used in psychological research (e.g. Luria, 1982). The main feature that we targeted with this item of the interview was formal categorization (TB21a), related to stabilization of meanings in theoretical thinking.

The given expressions could be classified according to a single discriminating feature (theoretical behavior, coded [1,0] on the feature TB21a), or they could be seen as a variety of unrelated objects, e.g. "this is a linear equation, this is a cubic polynomial, this is a formula for the area of a triangle, this one is complicated and I don't know what to do with it" (practical behavior, coded [0,1]).

The discriminating feature could be mathematical (e.g., degree, number of variables, being an equation or not). In this case the student's behavior would be considered as theoretical. If the chosen discriminating feature was "subjective", i.e. referring to the attitude of the speaker to the expression rather than to relations between concepts (e.g. "those that I can solve and those that I can't solve", or simple/complicated), the behavior would be considered as partly theoretical (because the classification was based on a single feature) and partly practical (because the discriminating feature was not related to relations between concepts) (behavior coded by [1,1]).

Two other features of theoretical thinking could also be expressed in the students' behavior in this question, both related to analytic thinking:

- interpretation of letters in algebra as representing variables rather than names for fixed classes of objects (TB321a), and
- rigorous interpretation of algebraic expressions (TB311a).

We explain these features of behavior in more detail. Some students may single out the expression "A = ah/2" as "not belonging" for some reason. In case their verbal behavior implied that, had the expression been written in the form y = ax/2, they would not have classified it as not belonging, we would assume that their thinking was practical. We assumed that a theoretically thinking student interprets letters in formulas as representing variables whose domains are conventionally defined, whereas the practically thinking student interprets letters as names for fixed domains of objects. For example, "A = ah/2" would be interpreted as a formula for the area of a triangle by a practically thinking student, with "A" being a name for the area of a triangle, "a" for the base and "h" for the altitude. The latter behavior would be represented by the vector [0,1] with respect to the feature TB321a.

Some students may classify the cubic expression $2x - x^3 + 78$ as an "equation". Noticing that it stands out in the set of given expressions as *not* an equation would be considered as representing linguistic sensitivity to formal symbolic notations (TB311a). A practically thinking student might implicitly add on to the expression the background of a context in which he or she encountered this kind of expressions, namely the context of solving equations. A student may have some reasons for considering the cubic as an equation: for example, thinking in terms of a computer syntax (rather than textbook syntax), the cubic might appear, by default, as an equation $2x - x^3 + 78 = 0$. However, even in the computer language, the expression by itself would not be considered as an equation, but only in the context of a command like "solve". For example, in *Maple*, the command *solve(2x -* x^3 *+78, x)* would produce a solution to the equation $2x - x^3 + 78 = 0$. Thus, even with this "good reason" we would classify this behavior as symptomatic of practical behavior.

Analysis of students' behavior in Question 1. The interviewed students' behavior in Question 1 is represented in Table 1. In relation to the feature of formal categorization (TB21a) the students' responses could be classified as follows.

1. Classification according to a single feature
    1.1 The feature is mathematical, based on relations between concepts
        1.1.1 Key feature: equations / non equations
        1.1.2 Key feature: degree
        1.1.3 Key feature: number of variables
        1.1.4 Key feature: number of values that an expression yields
    1.2 The feature is "subjective", based on relation between self and the expressions
        1.2.1 Key feature: can easily solve / can't solve
        1.2.2 Key feature: simple / complicated
        1.2.3 Key feature: makes sense / doesn't make sense
2. Not a classification according to a single feature - Naming of the items

A few students produced responses falling into only one from the above categories, but most proposed several possible groupings. In only one case, these groupings formed levels of classification: O3 singled out the cubic as a non-equation, and divided the remaining expressions into equations involving polynomials with one variable ($x - 13 = x$, $2x + 3 = 7$) and equations involving polynomials with more variables ($y = x^2 - 4x + 1$, $A = ah/2$). He confessed that, at first sight, he saw $y = x^2 -4x + 1$ and $A = ah/2$ as both involving two variables because, in "$A = ah/2$", he interpreted "a" as representing a constant. He said, that "in mathematics, letters such as a, b, and c are always taken for constants".

The student V3 first formed three groups: Group I: degree 3: $2x - x^3 + 78$; Group II: degree 2: $y = x^2 - 4x + 1$; Group III: degree 1: $2x + 3 = 7$, $x - 13 = x$, $A = ah/2$. The student explained that she put $A = ah/2$ into the group of first-degree expressions because she considered h as being *the* variable, but then she hesitated and changed her mind. She put it into the second group saying that she now considers a and h to be the variables.

The student O4 proposed three groups: Group I: no solutions: $x - 13 = x$; Group II: one solution: $2x + 3 = 7$; Group III: many solutions: $y = x^2 - 4x + 1$, $A = ah/2$, $2x - x^3 + 78$. It is possible that O4 thought of $2x - x^3 + 78$ as yielding many different values by substitution of numbers for x in the context of graphing the function $y = 2x - x^3 + 78$. In his rather messy explanation he used words such as "plug in a value", "vision", "graph", "any points will do":

AO: First you said it's not an equation and then you said it has many solutions. What where you thinking there?
O4: Well, it's sort of likeÉwhen you have aÉvision ofÉThere's no equals sign there, so, well, I guess, it doesn't say anything, because any points will doÉ
[É]
O4: Usually when I solve this kind of equation, you know, it's an equation of a certain plane, or y in this given by this, so I guess this one (points to $y = x^2 - 4x + 1$) has to do with a graph, and this one (points to $2x - x^3 + 78$), I don't know. It's harder to beÉ
[É]
AO: So what would you do with this? I mean what kind of question would you have withÉthe one you said is not an equation?
[É]
O4: Well, I guess, if you plug in a value, you get something, although it is not assigned to anything else, butÉ

Question 1. "Classification"						
	TB321a Interpretation of letters as variables		TB311a Rigorous interpretation of algebraic expressions		TB21a Formal categorization	
O1	0	1	0	1	0	1
O2	0	1	1	0	0	1
O3	1	0	1	0	1	0
O4	1	0	0	1	1	0
V1	1	0	0	1	1	1
V2	1	0	1	0	0	1
V3	1	0	0	1	1	0
V4	1	0	1	0	1	0
S1	0	1	0	1	1	1
S2	1	0	0	1	1	0
S3	0	1	1	0	1	0
S4	1	0	1	0	1	0
N1	1	0	0	1	1	1
N2	1	0	0	1	1	0

Table 1. Students' behavior in Question 1. Only three students behaved in a consistently theoretical way on all identified features.

It seems that O4 was not thinking in terms of equations and their solutions but rather in terms of expressions and substituting values for the variables in the expressions: the number of values one can plug in and obtain something sensible. He was classifying expressions with variables and his discriminating feature was the number of values for which they made sense.

Students O1 and O2 did not propose any kind of systemic categorization. O1 had four groups, naming them as follows: "This is nothing" (x — 13 = x); "straight line" (2x + 3 = 7); "parabola" or "a curvy graph" ($y = x^2 - 4x + 1$, $2x - x^3 + 78$); "area" (A = ah/2). O2 did not group the items in any way; he just named them in a similar way as O1, except that "A = ah/2" was described as "This, I don't know what it is".

Question 2. "Linear independence definition"

The second question asked the students to comment on a statement claimed by a fictional student to be a definition of the linear independence of vectors. The question was written as follows:

> *At a final exam in linear algebra, there was a question about the definition of linear independence of vectors. One student wrote:*
>
> A set of vectors $\{v_1, \ldots, v_n\}$ is called linearly independent if $a_1v_1 + \ldots + a_nv_n = 0$.
>
> *(i) What do you think about this definition?*
> *(ii) The same student then wrote that the set of vectors {(1, 2), (2, 4)} is linearly independent. Would you say that what the student said followed from his definition?*

Analysis of Question 2. The "definition" given in part (i) is one that we have encountered in our experience as teachers of linear algebra and we have analyzed it in one of our previous studies on students' difficulties (Sierpinska, 1997). Students saying or writing this kind of "definitions" have no feeling of incompleteness, because they read the equation $a_1v_1 + \ldots + a_nv_n = 0$ not as a subject of a verb but as a complete clause with the predicate "is equal to", represented by the equal sign.

The statement in part (i) is incomplete in two aspects: it does not specify (a) what the variables $a_1, \ldots, a_n$ stand for, and (b) what is the condition that these variables are supposed to satisfy. We expected theoretically thinking students to notice these two missing elements. Noticing (a) would be an expression, for us, of the interpretation of letters as variables (TB321a). Noticing (b) would be an expression of a hypothetical approach to mathematical statements (TB23a). We explain hereafter what we mean by that.

Some students may fail to notice that the cited "definition" contains no conditions on the variables and therefore cannot be used as a premise in the implication suggested in part (ii). Such could be the approach of students for whom the equation $a_1v_1 + \ldots + a_nv_n = 0$ in the definition of linear dependence/independence was not part of a condition on a linear combination of vectors, but an exercise: an equation to be solved. These students would answer the question (ii) in the positive, justifying it by some concrete solution to the equation $a_1(1,2) + a_2(2,4) = \mathbf{0}$. On the other hand, a student who would be aware of the conditional character of the *definiens* would have doubts about whether one can say that the cited "definition" implies the statement "the set {(1,2), (2,4)} is linearly independent". He or she might say that it depends on how one interprets the "definition". For example, if the equation is assumed to be true for *some* choice of the scalars a_i, then any set of vectors can be proved to be linearly independent by that definition, because, obviously, the choice of all scalars equal to zero would make the equation true. If, on the other hand, the equation is assumed to hold for *any* choice of the scalars, then only sets of zero vectors could be proved to be linearly independent.

Analysis of students' behavior in Question 2. Ten out of the 14 students made remarks about the lack of specification of the domains of variables in the cited "definition", and their behavior with respect to the feature TB321a was coded [1,0]. The other students appeared to take it for granted that a_i's represented scalars and v_i's — vectors, and their responses were coded [0,1].

Question 2. "Linear independence definition"				
	TB321a Interpretation of letters as variables		TB23a Hypothetical approach to statements	
O1	0	1	0	1
O2	0	1	0	1
O3	1	0	1	0
O4	1	0	0	1
V1	1	0	1	0
V2	1	0	0	1
V3	1	0	1	0
V4	1	0	0	1
S1	0	1	1	0
S2	1	0	0	1
S3	0	1	0	1
S4	1	0	1	0
N1	1	0	1	0
N2	1	0	1	0

Table 2. Students' behavior in Question 2. Six students behaved in a consistently theoretical way.

There were two categories of responses to Question 2(ii): responses containing conditional statements (coded [1,0] on TB23a) and responses without conditional statements ([0,1]). Seven students' responses fell into the first category, expressing doubts about the statement in (ii) being logically implied by the "definition" in (i). These students tended to speculate about the possible consequences of the "definition" rather than said that (ii) unconditionally follows from (i). However, all but one of these students also provided some concrete solution to the equation a(1,2) + b(2,4) = (0,0). This singular student was O3 who did not propose any solution to the equation, insisting that no logical conclusion could be drawn from the incomplete statement. We awarded all seven students the score of [1,0] on the feature TB23a, although perhaps only O3 behaved in a decidedly theoretical way. We decided to give the others the benefit of the doubt.

There were three kinds of answers among the skeptical students:

1. YES, BUT: "Yes, but the student quoted in (i) didn't specify anything about the a_i, so anything can be proved from this definition". These students were then adding that the equation a[1,2] + b[2,4] = [0,0] has a solution or were giving a concrete solution.

2. KIND OF: "It kind of follows from what the quoted student thinks" (implying: "but maybe not from what he just wrote"), and a solution to the equation a[1,2] + b[2,4] = [0,0] was given.

3. NO: "It is impossible to make a logical link because the ÒdefinitionÓis not completeÉYou can always write a set of vectors as a linear combination and put it equal to zero; but this doesn't tell you anything"

The seven responses of the "unconditional" category were close to: "Yes, of course, because the equation a[1,2] + b[2,4] = [0,0] is satisfied by a pair of scalars" (and the scalars were provided).

Question 3. "Linear dependence typo"

The third interview question cited a test problem in which typographical mistakes were made:

*Let **u**, **v**, and **w** be vectors in a vector space V over **R**.*

*Show that the vectors **u - v**, **u - w**, and **v + w** are linearly dependent.*

The intention of the instructor who wrote the test was to have a set of vectors, which was unconditionally dependent, e.g. **u - v, w - u, v - w**. But, in the formulation given in the test, the set {**u - v**, **u - w**, **v + w**} is independent *for some* choices of the vectors **u**, **v** and **w** (it is enough that these vectors form a linearly independent set); therefore the implied conclusion that "the set **u - v**, **u - w**, and **v + w** is dependent for any set of initial vectors **u**, **v**, and **w**" is not true.

In the interview the students were asked to first describe orally how they would approach the problem, without writing anything. Only after they did that, would they be allowed to carry out their plan. They were expected to notice that the statement they were asked to "show" was false, either during the period of planning or at the stage of carrying out their plan. The purpose of asking them to plan ahead was to give their capacities of theoretical thinking a chance, by allowing them to take a distance to a task they were quite likely to perceive as routine and easy.

If the student noticed that there was something wrong with the formulation of the question, he or she would be asked to propose a way of "fixing" it.

Analysis of Question 3. Noticing that there was something wrong with the cited test question required, first of all, a certain familiarity with the unwritten convention of interpreting variables in mathematical statements as being under universal quantifiers, unless otherwise stated (TB322a). According to this convention, the test question would be interpreted as: "*Show that, for any vector space V over **R** and for any vectors **u**, **v**, and **w** in V, the vectors **u - v**, **u - w**, and **v + w** are linearly dependent*", or as "*Show that if V is any vector space over **R** and **u**, **v**, and **w** are any vectors in V, then the vectors **u - v**, **u - w**, and **v + w** are linearly dependent*". If the default quantifier was assumed to be the existential one, there would be no mistake in the formulation of the problem. Indeed, *for some* vectors **u**, **v**, and **w** in V, the vectors **u - v**, **u - w**, and **v + w** are linearly dependent (it is enough to take **u** = **0**, the zero vector).

These conventions of interpretation are not always explicitly taught to undergraduate students. However, some students seem to be especially sensitive to the quantification of variables and are able to guess the conventions of interpretation by attending to the conclusions that the teachers and the textbooks draw from such conditional mathematical statements.

Another condition for seeing the statement in the test question as false, was a "definitional" understanding of the notion of linear dependence of vectors (TB21b), rather than an understanding restricted to being able to recognize or test if a given set of n-tuples (for concrete n) is linearly dependent. The practically thinking student could behave as if the problem was not so much in the mathematical falsity of the statement, but in the impossibility of carrying out the task of verification of linear dependence of vectors that are not known or not computable in some way.

From the point of view of noticing that the statement in the test question was false, the definitional approach to meanings in mathematics was perhaps more important than not confusing the meanings of "linear dependence" and "linear independence". For the statement would still be false if "*are linearly dependent"* was replaced by "*are linearly independent*". However, according to our postulated model of theoretical thinking, one would expect sensitivity to mathematical terminology in theoretically thinking students, and thus a correct use of the terms, without confusing their meanings (TB312a).

Supposing that a student had the required sensitivity to the quantification of variables and a definitional approach to the meaning of the term "linear dependence", how would he or she come to a realization that the statement in the test question was false? Let us imagine a possible way of reasoning:

1. We want to show that
2. if **u**, **v**, and **w** are any vectors in a vector space V over **R**
3. then
4. there exist scalars a, b, c, not all zero, such that
5. a(**u - v**) + b(**u - w**) + c(**v** + **w**) = **0** (*)
6. But (*) is equivalent to
7. (a+b)**u** + (-a+c)**v** + (-b+c)**w** = **0** (**)
8. One possible solution to (**) is the trivial one, with all coefficients equal to 0:
9. a + b = 0, -a + c = 0 and —b + c = 0 (***)
10. This implies that
11. a = b = c = 0
12. i.e. the system (***) has only the trivial solution.
13. Thus the trivial solution of (**) does not provide us with a non-trivial solution of (*).
14. Would some non-trivial solution of (**) provide us with a non-trivial solution of (*)?
15. Probably yes, but for such a non-trivial solution of (**) to exist, the set {**u**, **v**, **w**} would have to be linearly dependent.
16. If, on the other hand, this set was independent, then (***) would be the only solution, and since the only solution to (***) is the trivial solution a = 0, b = 0, c = 0, then also the set {**u** — **v**, **u** — **w**, **v** + **w**} would be linearly independent. Thus, this set of combinations is not unconditionally independent, as the problem requires to show.

Discussing the problem by referring to the definition of linear dependence (lines 1 — 5 above) would be, for us, an expression of a definitional approach to meanings in algebraic contexts (TB21b). Reasoning as in lines 8, 13-14 would be an expression of a proving activity (TB22a): making conceptual connections rather than following routine procedures. Speculations as in lines 15-16 would be an expression of hypothetical thinking (TB23a).

Reference to the definition of linear dependence could reveal students' sensitivity to logical connectives (TB322b). Possible mistakes could include saying, for example, "linear dependence means that if a(**u - v**) + b(**u - w**) + c(**v** + **w**) = **0** *then* not all a, b and c are zero". This kind of behavior could be interpreted as lack of sensitivity to the syntax of mathematical language.

The reasoning in line 8 is not based on formal logical deduction from the premise that **u**, **v**, and **w** are any vectors. Rather, it is based on an understanding of linear dependence and independence in terms of the existence of non-trivial solutions to a certain type of equation. This equation is homogeneous and thus always has a solution, namely the trivial solution with all coefficients equal to zero. What discriminates between linear dependence and independence of the set of vectors in this equation is the existence or non-existence of non-trivial solutions. In line 8 one does not look for all possible solutions to equation (*) but for a non-trivial solution. One tries to get this non-trivial solution by preparing to make an "informed guess", using the equations (***). In this case the attempt fails. But, had the combined vectors been different, for example, {**u — v, -u + w, v — w**}, equations similar to (***) could have left some of the coefficients free (e.g. a = b = c), leading to non-trivial solutions to the equation (*), and proving the statement true.

We expected that all students would approach the problem "practically", wanting to write the equations (***) whether consciously trying to make an "informed guess" on the basis of the property of homogeneous equations as outlined above or not. This was a familiar type of question and were the typos not made, the students would write the equations, *because this is how they would always do it* ([0, 1] on TB22a), solve the system of equations in a, b and c, obtain non-trivial solutions and stop there, or just write a line with something like "non-trivial solutions, so dependent". Most linear algebra students have a clear association in their minds between "non-trivial solution" and "dependent", whether or not they realize what are the variables and what are the constants in the equations they are considering, and what is supposed to be "dependent" (cf. Sierpinska, 1997). Instructors normally accept this kind of solution as correct, giving the students the benefit of the doubt, assuming that they made the missing conceptual links in their heads, even if they did not make them explicit on paper.

But in the present case, solving the system of equations in a, b, c leads to the trivial solution. A student who reasoned as in line 8 would conclude, at this point, that the chosen strategy was not effective (line 13) and would ask a question such as in line 14. But if the student did not consciously write the equations (***): a + b = 0, -a + c = 0, -b + c = 0 as *one possible* solution to the equation (**) (a+b)**u** + (-a+c)**v** + (-b+c)**w** = **0** then he or she may think that a = 0, b = 0, c = 0 is *the only possible* solution not to the system (***) but to the equation (**)! And then the student may believe that he or she proved the set of combined vectors to be independent.

If the student stops at this point and concludes that the way to fix the problem would be to change the conclusion to "are linearly independent", he or she would be assigned the score of [0, 1] with respect to hypothetical thinking (TB23a). Hypothetical thinking plays a crucial role in the reasoning outlined in lines 15-16. It is no longer enough to just draw conclusions, step by step. It is necessary to examine the validity of a chain of implications, and make explicit the assumptions made in each link. This reasoning requires understanding mathematical statements as conditional statements (if something holds then something else holds), whose truth or falsity refers to the logical soundness of the implication (i.e. the implicative relationship between the premise and the conclusion) and not to the factual truth or falsity of the premise and/or conclusion. The practically thinking student who concluded that he or she proved the combined vectors to be independent might say that, maybe, if the initial vectors were dependent, then the combined vectors could be proved to be also dependent. But it would not be an expression of hypothetical thinking if the student believed that he or she would have to *know* or establish it as a *fact* that the initial vectors were dependent. The theoretical thinker would want to establish which condition on the initial vectors *implies* which condition on the combined vectors.

Asking the students to explain how they would "fix" the test question was aimed at further revealing the way in which they reasoned in their attempts to solve the test question.

This question provides a good illustration of the role of pratical thinking as an *epistemological* obstacle to theoretical thinking, i.e. as a way of thinking against which theoretical thinking is built, and therefore without which it could not be built.

A practical thinker would immediately get down to solve the problem by means of some "usual technique", writing down an equation, setting the coefficients to zero, etc. Without practical thinking one might stay paralyzed in front of the exercise, overwhelmed by the different possibilities and their consequences (the possible conditions on the three vectors and their implications). Having undertaken the action "as usual", on the other hand, leads to an outcome which may be surprising and thus trigger off theoretical reflection. This reflection is already somehow structured or guided by the outcome of the calculation.

Relying on practical thinking alone could be paralyzing as well. If the solver has never asked him or herself questions about why the equation should be written and what is the purpose of equating the coefficients to zero, he or she may be unable to cope with the unexpected outcome, or not even consider the outcome as unexpected. The practical thinker is efficient in standard situations, but unable to act if there is anything that deviates a little from the expected.

Analysis of students' behavior in Question 3. Most students reasoned in ways we predicted in the above analysis, displaying behavior whose elements could be matched somehow with the lines 1-16 of the expected reasoning. Matched — from the point of view of the structure or pattern, not necessarily in meaning. We describe the behaviors of students O1 and O3, to illustrate what we mean by that.

Student O1's behavior	**Line of expected reasoning**	**TB feature**	**Value of TB**
"I would use the definition of linear dependence"	1	TB21b (def)	[1,0]
		TB312a (ter)	[1,0]
"So linear dependence depends on $a_1 v_1 + a_2 v_2 + a_3 v_3 = 0$"	5		
"and"	3	TB322b (log)	[0,1]
"one of the scalars is not equal to zeroÕ	4	TB312a (ter)	[1,0]
$C_1 (u - v) + C_2 (u - w) + C_3 (v + w) = z$	7		
"I'm just writing the equation, as always"	8	TB22a (prov)	[0,1]
$C_1 u - C_1 v + C_2 u - C_2 w + C_3 v + C_3 w = (0,0,0)$ $u (C_1 + C_2) + v (-C_1 + C_3) + w (-C_2 + C_3) = O$ (there was an arrow on top of the ÔOÕ)	7	TB23a (hyp) O1 did not notice that it was not assumed that $V = R^3$ PB311a (rig)	[0,1] [0,1]
$C_1 + C_2 = 0$ $-C_1 + C_3 = 0$ $-C_2 + C_3 = 0$.	9		
"They're all zero"	11		
"I'm finding u-v and u-w and v+w are linearly independent"	13	TB22a (prov)	[0,1]
"But I think I'm on the wrong track and I don't know what to do. Things are incontrollable, that's why I am confused, becauseÉ	14		
"ÉI <u>don't know if</u> these original ones are linearly dependent or independent set. I want to <u>find out</u> first if u, v, w are independent or independent so I can solve for that"	15-16	TB23a (hyp)	[0,1]

Student O1 was using a certain discursive style of saying things and writing things (not very rigorously, though), but he was not using the underlying style of thinking: he was not reasoning theoretically, he was following a flow chart of actions and decision boxes. At the decision box "are the coefficients all zero?" he was expecting the answer, "No", but obtained "Yes" and did not know what to do next.

The approach displayed by O3 could similarly be represented as following the pattern of our expected solutions, starting with a definition, writing the equation, solving it for the coefficients, getting puzzled after having solved it, trying to resolve the puzzle. But he was puzzled for a different reason and proposed to solve the puzzle in a different way, more in line with our model of theoretical thinking. However, student O3 was solving a different problem! He misread the question as asking to show that the set is linearly independent, not dependent. He noticed this eventually, quite late in the process of discussing the question, but was not much concerned about it. The given problem was flawed, anyway. The way he proposed to fix it was to add the assumption that the vectors u, v and w are linearly independent and ask to show that the combined vectors are linearly independent as well. Misreading the question did not imply that the student was not sensitive to mathematical terminology. In fact, Student O3 was very articulate and precise in using mathematical terminology. The way he used the definition of linear independence implied his sensitivity to both the quantification of variables and the logical connective (implication) involved in it. O3 was not really "puzzled" at obtaining the zero values for the coefficients, because this is what he expected, in the light of his reading of the problem. Rather, he suddenly noticed that he had been solving the problem mechanically, without checking if he had the grounds for writing the equations as he did. From then on he engaged in proving activity. He realized that he is missing an assumption about the dependence or independence of the initial set of vectors. At the beginning, he wanted to *know*, like O1, whether this set is dependent or independent. But, eventually, he switched to hypothetical thinking and re-formulated his conclusions in hypothetical terms.

Student O3's behavior	**Line of expected reasoning**	**TB feature**	**Value of TB**
"I would write it as in the previous definition"	1	TB21b (def)	[1,0]
"write it as a linear combination with scalar coefficients. Then, let it equal to zero"	5	TB312a (ter)	[1,0]
"Then, try to solve with the coefficients"	9		
"and see if the condition can <u>only</u> be satisfied if the scalars are in fact zero"	4	TB322b (log)	[1,0]
a(u-v)+b(u-w)+c(v+w)=0, <u>where a, b, c ∈ R</u>	5	TB311a (rig)	[1,0]
"I wrote it as a linear combination"			
au — av + bu — bw + cv + cw = 0	7	TB312a (ter)	[1,0]

"I distributed the scalar coefficients"			
(a+b)u + (c-a)v + (c-b)w = 0			
"I rewrote it factoring out the vectors"			
a + b = 0 c — a = 0 c — b = 0	9		
(solves using rref of the matrix representation)			
a = 0 b = 0 c = 0	11		
"I solved for the coefficients to see if they are zero"	12		
"(Silence) I put them equal to zero becauseÉehmÉIÉÉ see thatÉActually, actually, I thinkÉwouldn't we need something else?"	13-14	TB22a (prov)	[1,0]
"Some kind ofÉindications saying thatÉprobably, probably we need u, v, and w to be independent. I would need that to see in this formÉso that it can tell me that a + b = 0, c — a = 0, c — b = 0 and then solve."	16	TB22a (prov)	[1,0]
"So knowing that would allow to equate these coefficients to zero. If I don't know that, I can't"	16	TB23a (hyp)	[0,1]
"because, probably, one of these u minus another, etc., can make zero, also"	14	TB22a (prov)	[1,0]
"So this is not necessarily zero unless we have this condition here. (Silence) Okay, I have it. If u, v and w are independent, in order, to make this combination equal to zero, the coefficient a must be equal to 0, the coefficient b must be equal to 0 and c must be equal to zero. Thus, if we look at the original equation here, then we could see that we can only write this linear combination if the coefficients are 0, which means that they are linearly independent."	16	TB23a (hyp) TB22a (prov)	[1,0] [1,0]

A remarkably different behavior could be observed in student O2. When asked to describe how he would approach the problem, student O2 said:

> OK. I'll try to use, I don't know how to say that, like, make that in a bracket, then, one substr... uh, I can, I can, step, no, I don't know how to say it, we make like, plus, er... plus or minus, by one vector to another one, then give another equation, let another bracket, another three vectors, and then use multiple of one plus or minus the other one to get another one, until they got the base, er... like the base, the base, which they can't go lower, then and the first vector, the first, like, I, how to say, like there is a bracket, one vector that'sÉnumbers, which have common, the first one should be one, and then solve all the others one to make it equal to zero, then, put, then, we look at the, at the second one, and, make it to one, and, solve the other one, until they cancel. (O2)

The interviewer could not understand what the student meant to say and asked for more explanations. She suggested, "So, you would solve a system of equations? Is that what you would do?" O2 denied it, saying "not really", and then tried to explain his intentions, using a numerical example.

Usually, u stands for something like, 1, 2, 3, like thisÉand vÉ4, 7, 1, like thisÉ Then I substitute, like IÉfind this out with theÉ2, 5, 1, and this will be like 7, 2, 1É and this will be somethingÉ Then, I'll go toÉthis, I make it 1, and this, 5/2, 1/2, then I use this to subtract this, maybe times 7 and minus 7 and plus this, it'll be zero, and I'll go on, and this will be another scalar, it'll be 7, or 2 and this, 2, 0, something, then I'll go, by using this one, and this 2 over 7, and then I substitute by thisÉYou make it zero or somethingÉ Then, make it to all zero until there is the last one cannot be... can't be one, or cannot be substituted, right, or cannot be, er... cannot go lower than to see if the... if there's one of them is all zero, then this is dependent, if this is not zer... if... they... if the last step they one of them is like there is not all zero, then it is linearly independent. (O2)

This is what he wrote while saying the above:

(1, 2, 3) (4, 7, 1)

{u - v, u - w, v + w}

{(2, 5, 1), (7, 2, 1), (   )}

{(1, 5/2, 1/2), (0, 7, 2), (0,É)}

{(1, 0, -) (0, 1, 2/7) (0, 0 0, 0)

The student appeared to be giving an account of the procedure of row reduction, which he would normally use for checking if a concrete set of vectors given by their coordinates is linearly dependent or independent. But O2 was not using the terminology of row reduction; it was as if he could not speak the language of linear algebra at all. Maybe he was able to recognize certain terms in a question as pointing to the use of a certain kind of procedure, but he would not be able to formulate the question by himself, or explain it to another person without showing what to do in a particular example. This student did not solve the problem during the interview and never found out that it was ill formulated. He did not refer to a definition of linear dependence or independence. His notion of linear dependence seemed to be restricted to the notion of dependence of n-tuples, for some concrete n. It seemed also to be inseparable from the procedure of testing the dependence of n-tuples by row reduction of a matrix.

Let us now give an overview of the behavior of the group of students in this part of the interview.

Definitional approach to meanings (TB21b). Eleven students referred to a definition of linear dependence/independence in their approach to solving the test question; two more did so after some prompting. One student (O2) did not refer to a definition at all.

Proving activity (TB22a). One common feature of the behavior of the 14 students was that none of them justified his or her writing of the equations (***) by saying, "one possible solution to (**) is the trivial one". If they justified it at all, it was "on second thoughts" or after they had discovered that something was wrong with the question. The first attempt never contained any justification of this step. Their justifications later on referred to the added assumption of linear independence of the initial set of vectors. Then the equations (***) were understood as *the only possible* solution and not *one possible* solution. In mentioning the assumption "the initial vectors are linearly independent", students were thinking of it either as a way of "fixing the problem", or as part of a "reasoning by cases": Case 1: the initial vectors are independent; Case 2: the initial vectors are dependent. There were 4 students who ended up engaging in proving activity in discussing the question right after they obtained the unexpected trivial solution, 5 students who never got to this point, and 5 students who got there after repeated hints from the interviewer.

Hypothetical approach to statements (TB23a). Only 4 students' behavior bore clear indications of an awareness of the assumptions made in the question and made by the student him or herself during the solution. Student O3 was among these students. Student S1 was another student whose thinking we judged as hypothetical. Her first reaction to the statement of the problem was that it was "silly", because "*you cannot possibly take any three vectors, any three vectors at all, start mixing them and show something about them, because the vectors are so much of a choice that depending on your free choice, you can get them to be linearly dependent or independent".* She thus looked at the problem not, indeed, as a particular test question, but as an implication, a theorem. Without going into the details of the implication, she grasped the general pattern of the implication: If {**u**, **v**,É} is an arbitrary set of vectors in a vector space, then (a set of some linear combinations of the vectors **u**, **v**, É) is linearly dependent. Four other students were considered as not thinking hypothetically in the question. Student O2 was one of them; the other three were students who concluded that the combined vectors are unconditionally independent, and never changed their mind during the interview. The remaining six students were those who started with an unconditional statement of linear independence of the combined vectors, but who later stated their conclusion in a conditional form.

Sensitivity to specialized terminology (TB312a). Only six students were articulate in presenting their approaches to the problem and expressed themselves using correct terminology.

Sensitivity to logic (TB322a, b). Ten students used correct quantifiers in their definition of linear dependence/independence (although some of them interchanged the meanings of the two terms). The same ten students used also the logical connectives correctly in their definitions.

Question 3. "Linear dependence typo"												
	TB21b (def)		TB22a (prov)		TB23a (hyp)		TB312a (ter)		TB322a (∀)		TB322b (⇒)	
O1	1	0	0	1	0	1	1	0	1	0	1	0
O2	0	1	0	1	0	1	0	1	0	1	0	1
O3	1	0	1	0	1	0	1	0	1	0	1	0
O4	1	1	0	1	1	1	0	1	0	1	0	1
V1	1	0	1	1	1	1	1	0	0	1	0	1
V2	1	0	1	1	1	1	1	0	1	0	1	0
V3	1	0	1	1	1	1	0	1	1	0	1	0
V4	1	1	0	1	0	1	0	1	0	1	0	1
S1	1	0	1	0	1	0	1	0	1	0	1	0
S2	1	0	1	1	1	1	0	1	1	0	1	0
S3	1	0	1	0	1	0	0	1	1	0	1	0
S4	1	0	1	0	1	0	1	0	1	0	1	0
N1	1	0	0	1	0	1	0	1	1	0	1	0
N2	1	0	1	1	1	1	0	1	1	0	1	0

Table 3. Students' behavior in Question 3. Only three students behaved consistently theoretically on all identified features.

Question 4. "Log-log scales"

In Question 4 the students were shown two graphs in log-log base 2 scales (see Figure 2), both looking like straight lines and asked if they thought that these graphs represented linear functions. The upper graph could represent a linear function (namely y = 2x). The lower line could not represent a linear function because the rate of change of its values was not constant, given that f(4) = 2, f(16) = 4, and f(64) = 8.

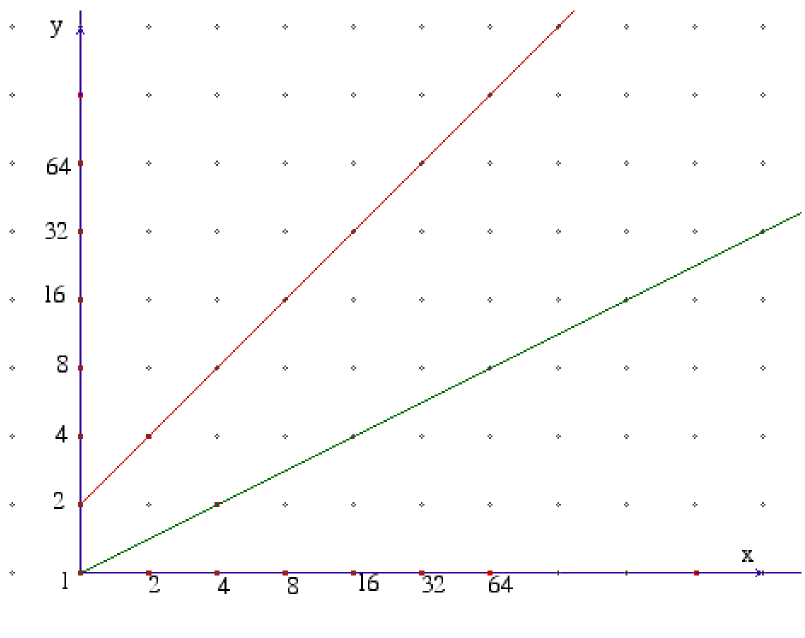

Figure 1: The figure accompanying Question 4, "log-log scales". The lower line was colored green and the upper line was red.

Analysis of Question 4. We expected the students to be puzzled by the unusual situation where a non-linear function was represented by a straight line and the axes were scaled in an uncommon way. We thought that a theoretically thinking student would want to investigate a little more this kind of representation, going beyond the question posed by the interviewer. In other words, we expected a "researcher's attitude" towards mathematical problems (TB1a) rather than an attitude of a student whose aim would be to just solve the exercise to the satisfaction of the instructor.

Students are so used to linear functions being represented by straight lines that they often believe that the converse is also true: a straight line necessarily represents a linear function. Some students do not analyze graphs as representations of relationships between variables (TB321b), but see them synthetically as shapes, sometimes even considering the axes as part of the graph of a function, and not as a"background" with a different status with respect to the relationship represented by the function.

Theoretically thinking students were expected to justify their answers by reference to some analytical characterization of linear functions, e.g. "a first degree polynomial function", or "a function whose derivative is constant" (TB21c "Definitional approach to meanings in graphical contexts").

Analysis of students' behavior in Question 4. Concerning the researcher's attitude (TB1a) we observed that four students (O2, O3, V2, V3) did not stop after having decided which one of the graphs represents a linear function but continued pondering the question. Here are some details about their behavior.

The student O2 verified the linearity of the functions by calculating difference quotients for some couples of points of the graphs. He then wondered, in relation to the lower graph, "Looks like a linear function, but it's not". He continued thinking about the representation, trying to visualize it, and making gestures with his hands to show the shape of the graph of the function if the usual scales were used. Now that he understood the non-obvious character of the representations, he proposed an interesting hypothesis. He conjectured something to the effect that lines above and below the red (upper) line should represent "parabolas", and the only line that could represent a linear function would be the one in the position of the red line. This is, of course, not true in general, but what counted for us, was the fact that O2 was making such bold conjectures. Indeed, the only straight lines in the log-log scales that represent linear functions are those with slope equal to one. Let $z = at + b$ be the equation of a straight line in log-log scales with base 2. This means that $z = \log(y)$, $t = \log(x)$ and $b = \log(2^b)$ where log stands for the base 2 log. Thus $\log(y) = \log(2^b x^a)$, whence $y = 2^b x^a$. This function is linear only if $a = 1$. So straight lines representing linear functions would all be parallel to the upper line in Figure 1.

The student O3 was very puzzled by the solution and started reflecting on it for himself, thinking out loud about the scales and generalizing this type of scaling. He started wondering what would happen if, instead of powers of 2, powers of 3 were taken: "If the y-intercept was taken at 3 would one still get this?"

The first reaction of the student V2 was that the functions are not linear but exponential, a conjecture that he immediately set to verify, by analyzing the relations represented by the marked points on the graphs, and calculating the difference quotients. As a result, he went back on his guess, concluding that "this [upper line] is y = 2x", and obtaining y = ˆx for the lower line. Then he started thinking about the reasons why one of the graphs represented a linear function and the other did not. He proposed that the upper line represented a linear function because "it is kind of, at 45 degrees", and because the same scaling was used on both axes.

The student V3 was quite focused on trying to understand the system of the scaling. She tried to figure out where would the zero be on the scale and discovered that, if one followed the same pattern, one would get 1/2, 1/4, 1/8, etc., always less and less and "0 would be nowhere"; "I don’t have the zero!" - she exclaimed.

The students’ behavior related to the interpretation of graphs of functions as representing relationships between variables (TB321b) was coded with [1,0] and [1,1] only. This implies a clear disposition to theoretical thinking in the group with respect to this feature. Even if, at first, some students claimed that both graphs represented linear functions because they were straight lines, they were able to engage in analytical thinking when prompted by the interviewer to "look at the scales". Here are some examples and details of the students’ behavior with respect to this feature.

Seven students’ first answers were that "both lines represent linear functions because the lines are straight" (for example, Student O1 said, "*Straight lines. Linear means straight, obviously, right?*"). But then, noticing that the interviewer was not passing to the next question, they would revise their answer (their behavior was thus coded [1,1]). Student O1’s next step, for example, was to characterize linear functions as "having constant slope", define the slope as the ratio $(y_2-y_1)/(x_2-x_1)$, and calculate this ratio in *one* interval.

In the other seven cases ([1,0]), however, the students would not be confused by the shape of the graphs. For example, Student V1 appeared to be instantaneously looking for the relationships between the coordinates of the marked points, and he claimed that the green (bottom) line was not a linear function because its formula "is $x = y^2$". The "red (upper) one seems to be, ’cause it’s simply 2x, y = 2x".

Only two students out of the 14 (O1, S3) appeared to believe that functions are not conceptually independent from the coordinate system in which they were drawn. O1, in particular, was extremely confused with the log-log axes, and could not accept them. He was saying,

> No, it does not work like that for a straight line, like…It'll always have to start at zero…[Here] it's just bad numbering…It just can't be that. You can't put a 1 there. It's not an axis then. It's then…If you put a 1 there, I can't even explain it, because no one has ever…It's just not a proper Cartesian plane. Or maybe it is, in a different way, but then you have no point (0,0) and nothing can be defined properly. (O1)

Seven students referred to some analytical characterization of linear functions from the start (behavior coded as [1,0] on the feature TB21c), and the other seven students did so after some prompting from the interviewer. Below are some examples.

The behavior of student S1 was assessed as representing [1,0] on TB21c. S1 first said "yes" [the graphs represent linear functions], but after a minute or so she said that "they don't", because "16 + 4 isn't 64". She was looking at the numbers marked on the x-axis, 1, 2, 4, 8, 16, 32, and 64, separated by equal distances. She was perhaps expecting constant rate of change in the numbers on the axis. She was making a statement not about the graphs but about the "distance function" on the axis. She was right that this function was not linear and was justifying it on the basis of a definition of linear function.

> Supposedly that the same scale is used, the linear equation is supposed to give a straight line, therefore the rise on one portion, would give the same rise on the other portion. (S1)

The interviewer realized that S1 was not talking about the graphs but about the axes, and must have looked very surprised. Then S1 noticed that she was on the wrong track. She begged for a break (she worked all day, and the interview started at 6 PM). The break was granted. The interviewer then made S1 read the values of the bottom function at 64 and 16 from the graph, and finally S1 started looking at the graphs rather than at the axes. Her notion of linearity of a function was that of "*the same slope uniformly all over the function"*, "*the rise on one portion would give the same rise on another portion*", "*if it's a linear function, the slope will be the same no matter where you are looking, at the whole function; if you are looking at the little bits and pieces of it, it's supposed to be the same everywhere".* She used this notion to verify if the upper line represented a linear function, but was making a mistake assuming that the axes started at 0 instead of 1. She obtained two different values for the slope and concluded that the upper line was not a linear function. This was corrected. However, she did not immediately conclude that the upper line was a linear function. She kept silent for a good while, after which she said, "*Yes, I guess they are*".

She declared herself not sure if her answers were right or wrong. She stressed the unusual character of the scales, not starting at zero. In total, she spent quite some time reflecting on the scales and trying to understand the system. She was then encouraged to verify in the same way if the bottom function was linear as well. She calculated the slopes in two intervals correctly and obtained different numbers. But she was not asked to verbalize a conclusion.

The behavior of the student O3 was coded as [1,1]. His first response, was *"Well, why not?*" and he mumbled *"straight"*, *"line"*. The interviewer then prompted him to look at the axes. He exclaimed, "*They're totally out of scaleÉnot growing properly"* . He further noticed that "*this isÉmultiplying by 2*". The interviewer then asked O3 if this discovery would change his answer, to which the student responded that he "*would have to check on the definition of linearÉfunctions* ". But he said he *"could not remember the definition of linear functions"* and proposed to "*re-scale the drawings, to see it"*. Carrying out his plan, and reading off the values from the lower graph, he found that it represents "*probably a function with a square root in itÉprobably half a parabola or something*". When he "re-scaled" the upper graph, he exclaimed, "*This is a line!*" and justified this statement by saying, "it hasÉx and y entries are always proportional".

Question 4. "Log-log scales"						
	TB1a (res)		TB21c (def-graf)		TB321b (graf-rel)	
O1	0	1	1	1	1	1
O2	1	0	1	1	1	1
O3	1	0	1	1	1	1
O4	0	1	1	1	1	1
V1	0	1	1	0	1	0
V2	1	0	1	0	1	0
V3	1	0	1	0	1	0
V4	0	1	1	1	1	1
S1	0	1	1	0	1	0
S2	0	1	1	1	1	0
S3	0	1	1	0	1	1
S4	0	1	1	0	1	1
N1	0	1	1	1	1	0
N2	0	1	1	0	1	0

Table 4. Students' behavior in Question 4. Only two students behaved consistently theoretically in this question.

Question 5. "Brillig numbers"

In the fifth question the students were given five statements about a set of integers called "brillig numbers" (odd prime + 2) and asked to pick one they would consider best suited for (a) a definition, (b) an explanation. They were also asked to tell if another statement about these numbers was true or false. Finally, the students were invited to express their opinions about the interest of the concept of brillig numbers and about what they found interesting in mathematics in general.

The text of the question was the following

> *Here are some true statements about brillig numbers. Read them carefully.*
>
> *Statement 1: Brillig numbers are, for example, 5, 7, 13.*
>
> *Statement 2: Any brillig number can be written as $p + 2$ where p is a prime number greater than 2.*
>
> *Statement 3: A brillig number is an integer number b for which there exists an odd prime p such that $b = p + 2$.*
>
> *Statement 4: All brillig numbers are odd.*
>
> *Statement 5: To obtain a brillig number, just take any odd prime number and add 2 to it.*

(a) Which of the above statements would you pick as a definition of the concept "brillig number"? Justify.
(b) Which one explains best what brillig numbers are? Justify.
(c) Is the sum of two brillig numbers a brillig number? Justify.
(d) Do you think that "brillig numbers" is an interesting concept? Could you name an interesting concept in mathematics? What is interesting in mathematics for you?

The word "brillig" was taken from the story, "Through the Looking Glass", by Lewis Carroll. The notion of "brillig number" was our invention.

Analysis of Question 5. This question was aimed at probing, mainly, the students' meta-linguistic sensitivity in mathematics: do they have a sense of the difference between definitions, and other elements of the mathematical discourse such as an example, a property, an explanation? The question did not go as far as addressing the students' awareness of the different kinds of definitions and their understanding of axiomatic definitions. Among the given statements one could only find definitions offering no more than a linguistic abbreviation.

Someone sensitive to the form of definitions (TB322c) could pick up Statement 3 as a first choice for a definition because it "sounded" like those statements that are named "definition" in textbooks: it introduced a new term and said what it was supposed to mean. He or she would have a sense that this statement was an equivalence: the term "brillig number" was equivalent to the phrase, "an integer number b for which there exists an odd prime p such that b = p + 2". Statement 5 could be chosen for a definition as well if the student only looked for an equivalence statement and ignored the style in which it was written, definitely not "bookish". In fact, there is nothing wrong with the style of this statement. Statement 5 could be classified as an "operational definition", one saying how to construct all brillig numbers (*Mala Encyklopedia Logiki*, 1988, p. 43). But sensitivity to the form of definitions would lead to a rejection of Statement 1 for providing only a sufficient but not a necessary condition: "if b = 5 or b = 7 or b = 13 then b is a brillig number". On the other hand, Statements 2 and 4 would be rejected for providing necessary but not sufficient conditions: "if b is a brillig number then it can be written as p + 2 where p is a prime number greater than 2"; "if b is a brillig number then it is odd".

Thus, answering question (a) requires a sense of a difference between an implication and an equivalence statement (TB2.5e). However, a higher sensitivity is needed for rejecting Statement 2 than for rejecting Statement 4. The fact that Statement 2 is a property is somewhat masked in the context of all five statements: it is known from Statement 3 that the necessary condition in Statement 2 is also a sufficient condition for a number to be brillig.

By asking question (b), we wanted to give a hint to the student that we do see a difference between a definition and an explanation in case he or she identified the two notions in answering question (a) and was really looking for the best explanation rather than for the best definition. In this case we were planning to go back to question (a) and allow the student to revise his or her answer.

In part (c) of the question we expected the students to answer in the negative and spontaneously justify their answer by giving a counter-example (e.g. 5 and 7 are brillig numbers but 5 + 7 is not because 12-2 is not prime, or because 12 is not odd), or by deriving a contradiction: all brillig numbers are odd, but the sum of two odd numbers is even, so the sum of two brillig numbers cannot be a brillig number (behavior coded [1,0] on the feature TB22b). An argument based on examples, e.g., "No, because 5+7 is 12 and 12 is not a brillig number because it is not listed in Statement 1" would be coded [0, 1] on the feature of refutation of a general statement by deriving a contradition.

In part (d) we expected the theoretically thinking students to display an appreciation of the intrinsic significance of mathematical concepts (TB1b) rather than a concern for their extrinsic significance such as, for example, their importance in examinations.

Analysis of students' behavior in Question 5. We start by describing the expression of the students' sensitivity to the form of definitions (TB322c) in the interviewed students. Students who chose Statement 3 as a definition often had no other justification for their choice beyond, "*it's just the way it's formedÉjust, more like a textual reference"* (O1). One student (S4) first chose Statement 5 on the ground that *"all the other statements are contained in it"*, but then said that Statement 3 would be a better definition because *"it contains more information*".

Concerning the awareness of the definition/explanation distinction, nine students appeared to possess this awareness. For example, student O1 chose Statement 3 as a definition and chose Statement 5 as an explanation in the pedagogical sense. Student O3 chose Statement 3 as both a good definition and a good explanation. His concerns were of a logical nature: he dismissed Statements 2 and 4 as only stating *"properties"* (his wording) thus implying that, for him, not only a definition but also an explanation should give both necessary and sufficient conditions. S1 rejected Statement 1, because "*three examples doesn't really define what a number isÉit doesn't exclude any of the real or imaginary numbers in any wayÉI mean you know that 5, 7, 13 are but you don't know what others aren't"*. She then said that Statement 2 would *"pass"*, but rather as "*something your teacher would sayÉan understanding of a definition*", because *"it's just the spokenÉexplanation of it"* . She did not explicitly state that Statement 2 did not provide a sufficient condition. But in analyzing Statement 3, which, for her, was a good candidate for a definition, ("*it describes the numbers well"*), she said, "*Is it trying to say that brillig numbers are only such thatÉOkay, it is saying that all brillig numbers are such that b = p + 2ÉYeah, I think that it's a definition, it can help you obtain the entire set of numbers b and it'll help you exclude all those that don't belong"*. S1 then rejected Statement 4: "*That doesn't define the brillig numbers. It says that all brillig numbers are odd, but does not say that all odd numbers are brillig. So it does not help to obtain the whole set"*. She found that Statement 5 said about the same as 2 or 3, but *"it's easier to understand if you are looking for a quick reference"*. Asked to pick the best explanation, S1 chose Statement 3, again, because, "*it’s the most rigid and leaves much less to interpretation and doubt than the others"*.

Students O2, O4, S2, N1 were confused when asked, "what would you pick as the best explanation". They seemed to believe that this is what they have already been asked before, when the "best candidate for a definition" was to be chosen.

Student S4 produced utterances, some of which appeared to imply that she made the distinction between definitions and explanations and others - that she did not. S4’s first answer to "pick a definition" was more like a response to "pick an explanation": she wanted to have a statement with the most information, so that *"you don’t have to deduce it by yourself".* She ended up point to Statement 3 as both a good definition and a good explanation.

Only four students noticed that Statement 2 was an implication and therefore could not be chosen as a definition (TB322d). Ten remaining students did not notice that but still nine would not choose the more obvious "ifÉthen" Statement 4 (*All brillig numbers are odd*) as a definition. Only one student (N1) chose Statement 4 as a definition. He insisted that Statement 4 characterizes brillig numbers completely, claiming that not only all brillig numbers are odd but also all odd numbers are brillig. Yet, when one of the observers pointed to him that 3 is an odd number, which is not brillig, he verified it using the condition in Statement 3. This convinced him that 3 is not a brillig number. As a result he amended his proposition, saying, *"All odd numbers greater than 3 are brillig"*, still claiming that Statement 4 is the best candidate for a definition. He also kept claiming that Statements 2, 3 and 5 were "saying the same thing", not seeing the logical difference between Statement 2 and the other two.

In question (c), all students (eventually) refuted the statement "the sum of two brillig numbers is a brillig number" in the expected "systemic" way (TB22b). The most popular (nine students) method of proof was by counterexample (O1, O4, V3, V4, S1, S2, S3, S3, N1). The five remaining students argued that since the sum of any two brillig numbers is even (as a sum of two odd numbers), it cannot be a brillig number (O2, O3, V1, V2, N2). We said "eventually", because not all students came up with a contradiction right away. Student O4 was a case in point. He started by looking at the general form of brillig numbers, put "p + 2 + q + 2 = (p + q + 2) + 2" and tried to prove that p + q + 2 is never an odd prime number, for p, q > 2. However, he could not find a convincing argument. (He never thought of proving that, since p and q are odd, then p + q + 2 is even). His first approach thus showed a logical weakness, since apparently, the negation of "always" was "never", for him: the negation of "the sum of two brillig numbers is a brillig number" was, "the sum of two brillig numbers is never a brillig number". As the interviewer started questionning O4 about the validity of his justification, he eventually got the idea of just coming down with a counterexample (5, 7, both brillig, their sum is 12, and 12 - 2 is not odd prime, 12 is not brillig).

Most students were quite articulate and willing to respond to question (d). Some students were proposing extra-mathematical and some intra-mathematical reasons for their appreciation of mathematical concepts. We attributed [0, 1] to extra-mathematical reasons and [1, 0] to intra-mathematical reasons. We counted arguments based on the use of concepts in the applications of mathematics as intra-mathematical reasons (thus including applications of mathematics into mathematics).

Below are some more details about the students' responses to question (d).

Extra-mathematical (not intrinsic) reasons for the appreciation of the significance of a mathematical concept. For example, the concept is considered relevant by the authorities in the field (e.g. is published in textbooks or renown journals); this appeared to be the idea of the student O1, who said that the notion of brillig number is "not interesting because I never heard of it up till now". Or, the concept is useful outside of mathematics, e.g. in pedagogy; for example, by being puzzling, it increases motivation to study mathematics (O3, S1, N1); or, by being simple and clear or understandable, it increases confidence in mathematics (S1, S2, O3).

Intra-mathematical (where "mathematics" includes applications of mathematics) or intrinsic reasons for the appreciation of the significance of a mathematical concept. Some students claimed that a concept is interesting if it is useful in the applications of mathematics, for example, in calculations (O2, O3, S3, S4), in financial mathematics (O1, S3), in cryptography (S4, N2). Or, if the concept is meaningful in the sense that results obtained in relation to it can be interpreted in terms of some concrete application and therefore one has some control over them (V1, S2, N1). Having interesting properties was mentioned by O3, V2, V3, and N2. For example, V3 said that she found brillig numbers an interesting concept, because it made her curious to find more about what can be done with these numbers and about their properties. Asked to name what kind of properties she would be looking for, V3 mentioned, "*if you take a brillig number and then still add 2 to it, if it would work always*". O3 and N2 said that brillig number is an interesting concept because it reduces to the concept of prime number which has interesting properties. On the other hand, some students (O2, V1, V2, S4) considered brillig numbers as *not* interesting because they could be reduced to prime numbers, and were therefore not introducing anything new to mathematics. Connectedness with many other concepts was mentioned as a condition of interest by V2, V3, V4, S1, S2; the ability to afford powerful syntheses and shortcuts was brought up by O3, O4, V4, S3, N2.

Thus most students mentioned more than one reason, and, in five cases, the reasons were both intrinsic and extrinsic. Nobody mentioned only extrinsic reasons.

Below we give more details about the students' views.

O2: "*Interesting in mathematics are things that you can* ***use*** *to solve numerical problems"*; "*There is certainly nothing interesting in linear algebra, because it's all proofs and no problems".*

O3: "*Interesting in mathematics is that, which makes people* ***think****, which is understandable and which has a great* ***use****". "Prime numbers are interesting because every number can be written as a product of prime numbers". "The way we used matrices to solve many systems of equations [was interesting in linear algebra]. But, probably, probably you would do mistakes when you were just solving them, by hand, like dozens and dozens of equations, but when you have a matrix, it's just so, like,* sealed togetherÉ*and you have less chances of making mistakes".*

O4 cited the ***power*** of the theory in Calculus to afford ***shortcuts*** in computing areas, volumes.

V1 named probability and statistics as interesting areas in mathematics because there "*you can obtain a result and you know what it means"*. He contrasted it with linear algebra, where "*it's harder to* ***see*** *what you are looking for"*. He said he preferred "*things that are not too abstract".*

V2: When the interviewer mentioned the twin primes conjecture, V2 was very interested in that. He said that he read the book on Fermat's Last Theorem by Singh. Asked what he found interesting in mathematics, V2 said that he liked "everything" but most of all the ***logic part*** and "*logic is something that all humans share".*

V3 said that, in general, what she found interesting in mathematics was the linking together of many concepts and theorems. She considered the concept of minimal polynomial (suggested by the observer) interesting because, from it, "*we can derive, let's say, the properties some theorem permits and it's like, all, you see* ***that everything is linked together***É.

V4 mentioned that what he liked about mathematics was that it's ***based on thinking*** rather than memorization, and that there are several ways of solving a problem. He considered minimal polynomials interesting because he found them ***powerful*** as shortcuts in solving problems.

S1: *"Pretty much anything can be turned into an interesting notion depending on how it is* ***presented*** *to the classÉ   For me, I would lose interest in the material that is just soÉout there. The way I study the material that I don't find I'm prepared to get to, I actually take the book, I take the reference books and I sit down by myself and I spend hours upon hours getting there, learning why and how. And I think a lot of students expect youÉUsually you walk into a class and you expect to deal with the material with the tools that you have been given in this particular class, whereas now the courses are beginning to ask you to* ***relate it*** *to other, external sources as well. And I find that there hasn't been one class where I have done well in where I'd have studied only by that book and I only went to class. I've always had to do extraÉbecause there always is the fine detail that you don't know. Somebody knows it, and you don't. (É) Because no matter how challenging and great [a concept is], and it could be so much fun and there is so much you can do with it, unless you* ***understand****, and unless you* ***feel comfortable*** *getting there and explaining it to yourself or somebody else, it's not a fun concept to play with in any way, it scares you off.* ***Cause mathematics can be a very scary thing****."*

S2 liked the notion of brillig numbers because it was clear and simple and inspired confidence about math. She appeared to need such experiences of understanding to make her feel "confident" and "secure" as a learner. She also expressed her need to "see" the general purpose of the concept she was learning. Unfortunately, in linear algebra she found it very difficult to "feel secure" and to get this sense of purpose. *"It [brillig numbers] sounds nice. IÉI, right now, canÕt see much of an application for it, but, I thinkÉthis may sound silly but itÕs the kind of thing thatÉthatÕs pretty clear and pretty simple, and might inspireÉlikeÉ* ***confidence*** *about math, because you can say, like, I know what a brillig number is! That is easy!É [Interesting in mathematics is] stuff that has a purposeÉLike, I donÕt know, IÉI did very well in linear algebra but I have to admit that I found it very difficult,* ***and the fact that the teacher used a test that was very similar to those in the past*** *so I could see the pattern of these questions, that was very* ***secure*** *for me, like, that made me feel betterÉ But the whole idea of linear algebra was veryÉout there? And I think, for me, I find something interesting when I can* ***see*** *it, and I know what itÕs talking about, and it makes sense to me. Like, if I could draw it, or if I could visualize it, and for linear algebra, I think, thatÕs a problem, because* *aside from these graphs that we had before, itÕs veryÉmind orientedÉitÕs verÉitÕs notÉ itÕs very conceptual but itÕs not practical and I prefer things that are practical, that I can see."*

Question 5. "Brillig numbers"								
	TB1b (intr sig)		TB322c (form def)		TB322d (implic)		TB22b (refutation)	
O1	1	1	1	0	1	0	1	0
O2	1	0	0	1	1	1	1	0
O3	1	1	1	0	1	0	1	0
O4	1	0	0	1	1	1	1	0
V1	1	0	1	0	1	1	1	0
V2	1	0	1	0	1	0	1	0
V3	1	0	1	0	1	1	1	0
V4	1	0	1	0	1	1	1	0
S1	1	1	1	0	1	0	1	0
S2	1	1	0	1	1	1	1	0
S3	1	0	1	0	1	1	1	0
S4	1	0	1	1	1	1	1	0
N1	1	1	0	1	0	1	1	0
N2	1	0	1	0	1	1	1	0

Table 5. Students' behavior in Question 5. Four students' behavior was consistently theoretical in this question.

S3: "*There's a lot you can do with the minimal polynomialÉyou can just take a big ugly matrix and reduce it to a diagonal matrix which is the ideal matrix, if you willÉIt's **useful**, it helps a lot*".

N1:"*I like statisticsÉ[because] when you are applying it, you knowÉI know what I am getting. If I get a result, I can interpret it, whereas in linear algebra you really can't*".

N2 said she liked abstract algebra and operations research most, because one gives the ***tools*** and the other uses these tools to solve application problems. She was also very interested in number theory for the purposes of applying it in cryptography.

Question 6. "Vorpal"

In the sixth question a four-element set was given, T = {1, 2, 3, 4}, in which an operation called "vorpal" was defined. The operation was defined by a Cayley table, and the symbol for "vorpal" was "∇":

∇	1	2	3	4
1	1	1	4	1
2	2	2	2	4
3	3	3	1	1
4	4	4	4	3

The students were told that,

This table reads as follows: vorpal of 1 and 1 is 1, vorpal of 1 and 2 is 1, vorpal of 1 and 3 is 4, vorpal of 3 and 4 is 1, etc.
We write:

$1 \nabla 1 = 1, \quad 1 \nabla 2 = 1, \qquad 1 \nabla 3 = 4, \qquad 3 \nabla 4 = 1$

and they were asked the following questions:

(a) Is vorpal a commutative operation?
(b) Does vorpal have a right-hand zero element?
(i.e. is there an element z in T such that for any x in T, $x \nabla z = x$ *?).*
(c) Define a left-hand zero element by analogy to the right-hand zero element. Then answer the question:
Does vorpal have a left-hand zero element?
(d) Try to define a different operation in the set T such that it has both a left-hand zero and a right-hand zero. Can you make these zero elements to be distinct, or is this not possible?

Analysis of Question 6. The elements of the structure <T, ∇> were denoted by the familiar arithmetic symbols of "1", "2", "3", "4", but the operation was none of the four arithmetic operations and it would be difficult to define it analytically with some formula involving the known operations. Moreover, the right neutral element for the operation was called a "zero" and the symbols "1" and "2" happened to be the right-hand zeros of the operation. In order to understand the definition and apply it correctly, one needed to disregard the familiar meanings of the symbols and terms and accept them as formal entities that had no other meanings than those implied by the definition of the operation (TB21b).

A correct application of the definition in part (b) required also a sensitivity to the quantification of variables and to the order of quantifiers (TB322a). This sensitivity was also necessary in defining the "left-hand zero" element, for the student had to notice the quantifiers, their nature and order in the definition of the right-hand zero element, to see what had to remain constant and what needed to be changed in the construction of the analogy. The student had to be able to correctly negate the quantified statements in proving that vorpal is not commutative and had no left-hand zero element.

Question (d) had two parts. First, the student was asked to define an operation in T, which would have both a left-hand and a right-hand zero element. This part would discriminate between theoretical and practical thinkers insofar as the students would reveal their understanding of the formal status of the symbols used to define the structure T with operation "vorpal" (TB21b).

In the second part of the question, we asked the students to prove the uniqueness of an object satisfying certain conditions: the impossibility of the existence of an operation with two distinct one-sided neutral elements. We decided that students who would attempt proving this impossibility starting from the defining properties of the left- and right-hand side zeros would be regarded as thinking theoretically with respect to the use of axiomatic definitions in reasoning, or engaging in "axiomatic reasoning" ([1, 0] on TB22c). An axiomatic reasoning could go along the following lines:

> Suppose there exists a left-hand zero, z_L, and a right-hand zero, z_R, for an operation ¥ in T;
> then, by definition, for all x in T,
> x ¥ z_R = x (1), and z_L ¥ x = x (2).
> In particular, for x = z_L , (1) implies z_L ¥ z_R = z_L (3) ;
> For x = z_R , (2) implies z_L ¥ z_R = z_R (4).
> Equalities (3) and (4) imply that $z_R = z_L$.

Analysis of students' behavior in Question 6. We refer the reader to the Table 6 for the students' behavior with respect to the definitional approach to meanings (TB21b) and sensitivity to quantifiers (TB322a) in their responses to questions (a)-(c). We discuss more in detail only the students' behavior in part (d).

In part (d) nobody came close to the "axiomatic reasoning" (TB22c) outlined above. Thus their behavior was coded [0,1] on TB22c. The students approached the problem by trying to construct the postulated object. Some students just declared themselves defeated, "I tried and tried and it didn't work. So I don't think this is possible". They did not attempt any formal or even systematic reasoning. Other students tried to systematically check all possible cases. A few others attempted formal reasoning. But even in the latter group, arguments were of a *local, arithmetic* type, specific to the context of the four-element set T and based on the property that if an operation in T has a right-hand zero element then the table of the operation has a column of [1, 2, 3, 4], and if it has a left-hand zero element then the table has a row of [1, 2, 3, 4].

A priori, a "local, arithmetic" but still systemic proof of the impossibility of having distinct one-sided zero elements could proceed by *contradiction* (call this an LAC proof, for "local arithmetic proof by contradiction"), or by proving the necessity of an equality of the two zeros (call this type of argument LAN, for "local arithmetic proof of the necessity"). More precisely,

> LAC: Suppose i is a lhz and j is a rhz and i › j. Then the i'th row is [1, 2, 3, 4] and the j'th column is [1, 2, 3, 4]. Therefore the number in the ij'th position would

have to be equal to j because i is a lhz element, and it would have to be equal to i because j is a rhz; but i ›j, a contradiction.

LAN: Suppose i is a lhz and j is a rhz. Then the i'th row is [1, 2, 3, 4] and the j'th column is [1, 2, 3, 4]. Therefore the number in the ij'th position is equal to j because i is a lhz element, and it is equal to i because j is a rhz. Hence i = j, and the lhz and the rhz are equal.

However, the LAC and LAN arguments as outlined above were not obtained from the students in this articulate form. The three students (O2, O3, V2) who came close to the LAC or the LAN argument fell short of articulating the contradiction or the necessity in a general form. We coded their behavior [0,1] on TB22c ("Axiomatic reasoning") and [1,0] on the feature TB22a ("Proving activity"). The argument of O3 was close to LAC; the arguments of O2 and V2 were a mixture of LAC and LAN. The argument of O2 was very poorly articulated.

Four students (V1, S1, N1, N2) came close to LAN, but did not formulate the necessity in an analytic form. We coded their behavior [1,1] on TB22a ('Proving activity"). These four students proceeded by "checking all cases", and expressing the impossibility of choosing another element for the other zero in each case by some form of gesture, accompanied by words such as "I can't put anything else here". We considered this type of response as a display of systematic but not fully systemic reasoning.

Six students (O4, V3, V4, S2, S3, S4) just stated the impossibility with conviction, although one of them (S3) tried hard (without success) to find some systemic argument. All these students' behavior was coded [0,1] on TB22a ("Proving activity").

The student O1 was not asked the question 6(d). However, knowing his confusion with even the definition of the right-hand zero element and his general reluctance to justify his statements, we assumed that his behavior in this question would have been practical, and coded his behavior [0,1] on the feature TB22a. We did not want to ignore this question altogether for not having the data from all the students, and we did not want to ignore student O1, either, because his responses to other questions were interesting. So we decided to speculate on his possible response in this case.

Here are some more detailed descriptions of the students' behavior.

After reading the question, O3 responded in the negative and immediately set to justify it. He was using the "theorem" about the correspondence between i being a rhz and i'th column being [1, 2, 3, 4], and j being a lhz and j'th row being [1, 2, 3, 4]. He said that, having the first column [1, 2, 3, 4], if he "moved" the row [1, 2, 3, 4] down from where it was in his first table (1st row) *he would lose the 1*.   But he said he was not sure about his argument, and then he laughed and said, "he thinks he is sure". Then he suddenly realized that he was not taking into account all possibilities (he only considered the case where 1 was the rhz, which could be a consequence of his fixation on multiplication: he called his operation "multiplication"). He realized that the symbols 1, 2, 3, 4 did not represent ordinary numbers ("*get rid of these numbers, like, they're arbitrary"*). He then checked the cases of the rhz being 2 or 3 or 4 and then confirmed the impossibility of having distinct left-hand and right-hand zeros.

V2 took a very long time to answer the first part of the question (d) (find an operation in T with both a left-hand and a right-hand zero), mainly because he thought he should define his operation (which he called "vorpal" again) by a formula. He found one, namely a ∇ b = |a-b| + 1, which does have, in T, both a lhz and a rhz (1 is such an element). He was quite amazed when told that one could write absolutely anything in the table and this would still define an operation in the set T. From there, V2 was then able to resolve the second part of the problem very quickly, using his "theorem" that the existence of a rhz element expresses itself by a column of [1, 2, 3, 4] and the existence of a lhz element expresses itself by a row of [1, 2, 3, 4].   His argument was a mixture of a forward proof of a logical necessity (LAN) and of a proof by contradiction (LAC): he first claimed that a rhz yields a column of [1, 2, 3, 4] and, if this happens to be the first column then the only way one could put a row of [1, 2, 3, 4] would have to be the first one, because the only place where there is 1, is the first row. If one puts the row somewhere else then one has to put two different numbers in one place. The articulation of the argument was not quite rigorous, and he did not try to formalize his proof.

The student O2 eventually came close to an argument by contradiction of type LAC. But he first came up with a table in which 2 was both the rhz and the lhz with all the remaining boxes empty. He might have thought that he thus solved the problem, by finding an operation where the zero was different from 1, but the interviewers were not making any approving noises, so O2 realized he didn't get an answer yet. After some thinking, he stated, "*I don't think it's possible"*. He did not spontaneously embark on justifying his claim, but did so when prompted. His argument could be reconstructed, approximately, as follows (it was very poorly articulated): if there is a rhz element then there is a column of [1, 2, 3, 4], and the element that appears on the diagonal must be what it is ("*this diagonal stuff is fixed*"): i.e., if the column is the j'th column then the jj'th element must be j . If the lhz is the same, then "*this is correct"*, because the element in the row [1, 2, 3, 4] which is on the diagonal is the same as that in the column [1, 2, 3, 4]. But if the lhz is not the same then "*it's wrong"*. The student was unable to articulate his argument in a clearer fashion.

The student V1 drew 4 tables filled only with one row and one column of [1, 2, 3, 4]. In the first table, 1^{st} column and 1^{st} row were filled, in the second table 2^{nd} column and 2^{nd} row were filled, etc., till 4^{th} column and 4^{th} row. The way he reasoned could be close to LAN or "systematic check", but V1 had trouble articulating his argument. He said, "*I don't think it's possibleÉ 'cause Éthere are four ways of doing itÉand all of themÉit's the sameÉthe same elementÉ'cause ÉI don't know how to say it"* . This argument would be close to LAN, if the student were able to formulate it explicitly.

S1 quickly figured out that, in order to have a rhz and a lhz element it is enough to put a column of 1,2,3,4 and a row of 1,2,3,4 and she drew a table with such first column and first row without filling in the rest of the table, but realized that that "*would be a right-hand and a left-hand zero, but it wouldn't be distinct, it would just be one".* She used the dollar sign for her operation. She was at a loss when attempting the second part of the question, and started speculating about what would happen if she changed the order of the numbers: she put them in the order 3, 4, 2, 1. When asked why she was changing the order, she explained she was doing it to get rid of the ordinary meaning attached to the symbols 1, 2, 3, 4 and thus make the situation more abstract,*" to break the routine"*, and stop thinking "*in a strict number way"*. She begged for some time to think on her own, and ended up with a reasoning close to LAN:

> Basically, what I just realized by writing out the two zeros that I'm looking for, x dollar sign z equals x, and z dollar sign x equal x. Basically, I realized that by having this number here, in the first column, be it the right-hand zero, it obliges, in the first position, next to 1, to have a 1, next to 2 to have a 2, and so forth. Therefore, if any of these four were to become a left-hand zero multiplier, they would, automatically, in the first position, be obliged to put one of their own up here. Therefore, the right-hand and left-hand zero multiplier would have to be the same. (S1)

N1 took about 15 seconds to figure out the solution, and he justified it going through the possible cases of the rhz being 1, 2 or 3 or 4, in terms of the columns being 1, 2, 3, 4 and arguing that the only way to have a 1, 2, 3, 4 row in each case would be to have that row with the same index as the column (a reasoning close to LAN):

> I don't think it is possible because to have right-hand zero you got your row has to start at 1, yeah, your row has to start at 1 and then has to go 1,2,3,4 and the only way that this could happen is in the first row but if you put in the first row, then your set would be the

same. Even if you shift this to the second row, then you have to have a 2 in the second element of the, I am sorry, if you shift this to the second column, you need a 2 in second element of this row so you can only have them here. Basically the elements on all of them have to be the same. So I don't think it is possible to have distinct zero elements. (N1)

N2 also produced a similar argument. Her reasoning was in terms of rows, going through the cases of having a row of 1, 2, 3, 4 and talking about the necessity of then having the corresponding column being the same. (Reasoning close to LAN)

The student V3 answered the first part of question (d) by taking the ordinary operation of multiplication as her example of an operation with both a lhz and a rhz element (number 1), disregarding or not being aware of what it means that the operation should be "in T": her table was simply a multiplication table for the 4 numbers 1, 2, 3, 4. She then tried an addition table again going beyond the set T. The interviewer, looking at her tables, asked, "Is this in T?" and she immediately understood what this means ("*it doesn't give me elements from T"*). She was then introduced to the possibility of defining operations through tables in finite sets. She accepted this notion as fun ("*So it's just numbers?*") and quickly solved the first part of the question. She filled only the first row and the first column of the table and did not bother filling in the rest ("*that would be anything"*). In trying to solve the second part of the question, she drew a table where the elements of T were listed horizontally in the order 2, 3, 1, 4 and vertically, in the order 3, 2, 1, 4.

¡	2	3	1	4
3	2	3	1	4
2		2		
1		1		
4		4		

Perhaps she thought that this was a way to get the row and the column corresponding to the rhz and the lhz elements not crossing on the diagonal and thus having them different, but she realized that they were both the same, equal to 3. After some more silent thinking, she declared, "*it has to be the same"*. She had to be prompted for a justification, and what she said was more an account of how she tried to get an example and it didn't work, not a mathematical argument (e.g. "*here it's not working"*, "*I cannot find distinct"*).

We end with an account of the behavior of S4. She spend much more time on this question than other students, and, unlike other students, appeared to be struggling (albeit unsuccessfully) to produce a general algebraic argument. She first appeared to take the operation "times 1". When it was clarified that the operation should be binary, S4 called her operation "y" and started filling her table with columns of 1, 2, 3, 4. By the time she had written down three such columns she was interrupted by the interviewer who told her that now she had 3 rhz elements. S4 uttered exclamations of surprise. She then went back to the definition and tried to formulate the condition for an operation to have a lhz and a rhz. She was saying and writing: "*z which is the zero element, must be equal to zx, must be equal to xz, must be equal to x"*. She was thus apparently assuming that z is a two-sided zero. The interviewer pointed out to her that the zeros are assumed one-sided and repeated the question, writing: $x\nabla z = z'\nabla x = x$ in S4's worksheet. S4 then argued "*times the number over itselfÉz over z.ÉIt will always give you x, right?*" Perhaps she meant to say that, if $x \nabla z = z'\nabla x = x$ then z' = z = x/x = 1. She seemed to implicitly assume that the given set of symbols 1, 2, 3, 4 was a subset of the real number system, with all its operations well defined and valid. She still appeared to want to "divide" these elements as numbers in the ordinary sense. Indeed, when explicitly asked to explain what would it mean to divide 3 by 3 in the sense of, for example, vorpal, S4 became rather impatient and told the interviewer that she was thinking of ordinary division in real numbers and that she was allowed to take it because she was not supposed to deal with vorpal anymore in question (d). The interviewer, after clarifying the question once more, said, "So please define your own operation". After a period of silence, S4 drew a table for an operation in T. She gave it a times sign "×", but then changed it to the delta sign. She filled in only the diagonal elements and made them 1, 2, 3, 4. When the interviewer insisted on knowing what is 2 delta 3, S4 responded, "*it doesn't existÉI leave it like this"*. The interviewer again insisted that this is not a finished definition. After some silence, she filled in the remaining boxes in the table, getting four identical columns of 1, 2, 3, 4.

The interviewer pointed to the fact that this operation had no lhz element because it had no row of 1, 2, 3, 4, and came back to the vorpal operation and explained how it had two rhz and no lhz. The question was then repeated again. S4 was astonished that she was supposed to get both zeros in the same operation. She then changed the first row in her table to 1, 2, 3, 4. The interviewer then concluded that delta had both a lhz and a rhz but they were the same and repeated the question about the possibility of making them distinct. After having thought for a good while S4 responded that it was not possible, but she did not justify her statement. As the interviewer decided to close this part of the interview, she asked, "So what is the correct answer?"

Question 6. "Vorpal"								
	TB21b (def)		TB22a (prov. activ.)		TB22c (axiom. reas)		TB322a (quantifiers)	
O1	1	1	0	1	0	1	1	1
O2	1	0	1	0	0	1	1	1
O3	1	0	1	0	0	1	1	0
O4	1	0	0	1	0	1	1	0
V1	1	0	1	1	0	1	1	0
V2	1	0	1	0	0	1	1	1
V3	1	0	0	1	0	1	1	1
V4	1	0	0	1	0	1	1	0
S1	1	0	1	1	0	1	1	0
S2	1	0	0	1	0	1	1	1
S3	1	0	0	1	0	1	1	0
S4	1	0	0	1	0	1	1	1
N1	1	0	1	1	0	1	1	1
N2	1	0	1	1	0	1	1	0

Table 6: Students' behavior in Question 6. No student behaved in a consistently theoretical way in this question.

Question 7. "Beliefs"

Our discussion of this question will take more space than in the previous questions. In will be composed of two sections. In the first section we will give the rationale and formulate our expectations of students' behavior. In the second one we will discuss the students' actual behavior in the question, aiming at establishing an "epistemological profile of the group of high achievers".

There were four parts in this question, expecting to reveal the students' attitudes towards 1) truth, 2) mathematical knowledge, 3) study of mathematics, and 4) mathematical proof. In each part, the student was presented with several "personal statements" and asked to choose one that appeared to describe best his or her own attitude or, if none of the given ones fitted, to use his or her own words to answer the question. The "personal statements" were written so as to reflect the epistemological positions of "silence", "received knowledge", "subjective knowledge", "procedural knowledge" and "constructed knowledge" as proposed by Belenky, Clinchy, Goldberger, Tarule (1997). We were assuming that the position of "constructed knowledge" came the closest to the characteristics of theoretical thinking.

Below we explain in more detail where we think the positions of "silence", "received knowledge", "subjective knowledge", "procedural knowledge" and "constructed knowledge" stand with respect to the use of theoretical thinking in learning mathematics.

"Silence"

"Silence" is described by Belenky et al. (1997) as the position of those who, in their own eyes, know nothing and can learn nothing.

Reflective thinking: Silent knowers' attitude is unreflective: they do what they are told or shown what to do.

Systemic thinking: Knowledge is not even an aggregate of ideas for them; it is rather some kind of implicit "know-how"; they try to imitate the actions of someone who would show them what to do. They do not occupy themselves with evaluating statements as "true-false", or "valid-invalid"; their concern is with actions, some of which may "work", while others may not. But Silent knowers don't think it is up to them to decide what works and what does not. They will perform actions that will not work if this is what they think is expected of them. Silent knowers do not hypothesize: they accept things as they are and as they come.

Analytic thinking: Silent knowers use language at the most basic level of everyday social interactions. They don't believe they have anything important to say.

Thus the position of "Silence" does not intersect with theoretical thinking.

"Received knowledge"

Reflective thinking: People in the position of Received knowledge "collect facts but do not develop opinions" (Belenky et al., ibid, p. 42). Knowledge is equated with "accepted wisdom" or "expert wisdom". It is a citation from an authority, not a conclusion from one's own thinking.

Systemic thinking: Knowledge is seen as a collection of facts to memorize and procedures to be carried out. A Received knower is more likely to study science and mathematics than humanities: she believes that "there are absolutes in math and sciencesÉ[whereas] work in other courses seems to accomplish nothing. It doesn't really matter whether you are right or wrong, because there isn't a right or wrong" (ibid, p. 42). While Received knowers are concerned about knowing "the truth", they are not concerned with validation of knowledge because, for them, truth is warranted by the authority of its source. They do want to make sure they have to do with an authority. A book, a printed and published text is an authority. They believe that "teachers are always more or less rightÉbecause they have books to look at" (ibid, p. 39). But they would become indignant upon learning that a teacher based her lecture on her own research and not on "books". Without the stamp of some superior authority this would be just some "made up knowledge", unworthy of their attention. For Received knowers scientific knowledge is not hypothetical: it is supposed to be certain and provide one with sure "facts". There is no gradation of truth and the existence of contradictory theories is inconceivable (Belenky et al., ibid, p. 41).

Analytic thinking: Received knowers use language as idiom and do not analyze its structure or the meaning of its components. They do believe, however, in the power of words for transmitting knowledge (Belenky et al., ibid, p. 36).

Thus, like the position of Silence, also Received knowledge is contradictory with theoretical thinking.

"Subjective knowledge"

Reflective thinking: Subjective knowledge is equated with personal opinion and/or experience. Subjective knowers may be interested in knowing for the sake of knowing and be quite creative in expressing themselves.

Systemic thinking: Subjective knowers do not distance themselves from the products of their thinking; they identify themselves with them. They do not compare systems of concepts although they might compare their own attitudes with attitudes of others. Thus, the thinking of persons in this position is not systemic; it is contextual and situational. Statements, which might be considered as contradictory from a systemic point of view, are regarded as resulting naturally from differences in interpretation or from the context of their utterance. The role of reasoning is taken over by stories of the circumstances in which one's own experience has led one to a conclusion. For persons in this position, concepts are linked not by logical links and relations of generality among categories of thought but by belonging to the same sphere of practice which is shared by a community who uses them in the same way, in the same kind of expressions and contexts. Thus the best way to explain a concept is through examples of its use.

Subjective knowers do not consider science as providing some kind of truth about or even a viable model of reality. The notion of "fact" has little sense for them. There are many "truths" which are different for different people. No one is absolutely right or absolutely wrong, "so I could be right, too". People in this position trust their own intuition or "gut feeling". This "relativity" of truth does not, however, result in the construction of "hypothetical knowledge". Subjective knowers consider their own knowledge as absolutely true for them, and they do not seek the identification of the assumptions under which they make their decisions.

Analytic thinking: Subjective knowers' relation to the objects of their thinking is direct, experiential. They may use language to express their thoughts about something, but they would not admit that they are talking about the meaning of words and not about this something. Thus their thinking is synthetic rather than analytic.

Subjective knowers, therefore, share with theoretical thinkers their reflectivity, and their concern with meaning and justification, but they are not concerned with formal representations of these meanings and justifications. We could say that Subjective knowers are not so much "practical knowers" as "intuitive knowers", in the sense of Fischbein (1987).

"Procedural Knowledge"

Reflective thinking: Like the subjective knowers, procedural knowers are capable of engaging with thinking for its own sake. However, unlike them, procedural knowers do not identify themselves with the products of their thinking and are able to take a distance to knowledge.

Systemic thinking: Procedural knowers' thinking is systemic. Procedural knowers see knowledge as built on logically interrelated systems of concepts and procedures. They do not trust their intuitions and "gut feeling". But their thinking may become inflexibly formal, clinging to a set of rules without distinguishing between the relevant and the irrelevant rules (an extreme example is that of high school students who think that writing the solution of an equation on a new line is as important as the rule that adding the same number to both sides of the equation leads to an equivalent equation). Procedural knowers don't think of themselves as owners of the knowledge they apply. This knowledge belongs to experts, who are responsible for the validity of this knowledge. Even if they "engage in conscious, deliberate, systematic analysis" (ibid, p. 93), Procedural knowers do so by following standard procedures that they have learned from the experts. They do not question these procedures. Using the terminology of Chevallard, they work at the level of *techniques* to solve standard tasks (Bosch & Chevallard, 1999, p. 84-86); they do not engage in constructing *technologies* and *theories.* For example, they want to know how to solve a type of equation but are not engaging in developing a discourse to describe and justify the technique they are using (a part of technology) nor, a fortiori, in embedding this discourse in a more general theory about the tasks, techniques of their solution and the technology. The latter would make them see that other techniques and approaches to solving their problem could be used or even that the problem could be posed differently. Belenky et al. (ibid.) say that in procedural knowers, "form predominates over content" (p. 95) and that this attitude may sometimes degenerate into *methodolatry*, whereby a person believes that research methodologies are automata for producing scientific truth and that, therefore, science should preoccupy itself mainly with the development of methodologies, never mind the relevance of the results that these methodologies could produce. Procedural knowers do not consider scientific knowledge as hypothetical or relative to a set of assumptions because they interpret the assumptions of a procedure as part of its "instructions for use".

Analytic thinking: Procedural knowers regard language as a means of representation of knowledge and they believe that it is possible to invent representations that are unambiguous and clear. They are sensitive to the formal and conventional aspects of scientific discourse.

Procedural knowers' thinking is, therefore, reflective and analytic, and systemic without being hypothetical. A Procedural thinker could be a theoretical thinker who turned the theoretical framework of his or her research into a dogma.

"Constructed knowledge"

The last category of epistemological positions proposed by Belenky et al. (ibid.) is that of "constructed knowledge".

Reflective thinking: Like Subjective and Procedural knowers, Constructed knowers see a purpose in knowing for the sake of knowing.

Systemic thinking: For Constructed knowers, knowledge is composed of systems of logically interconnected concepts, statements and procedures. So it is for Procedural knowers, but, while the latter accept the validity of these systems upon the word of the experts, Constructed knowers demand a proof of this validity, so that they can judge for themselves. Aware of the human (as opposed to "revealed") source of knowledge, Constructed knowers admit its hypothetical character, relative to explicit as well as implicit assumptions.

Analytic thinking: Constructed knowers consider themselves as participants in the human construction of knowledge, as actual or potential partners of "experts", and not, like Procedural knowers, only just the "users" of the expert knowledge. Knowledge is seen as constructed in the minds of individuals and shared through "conversations" among partners rather than "delivered" in a one-way fashion from the person who knows to the persons who do not. Individuals construct knowledge based on their own experience and intuitions as well as on reasoning according to conventional rules, and therefore linguistic representations of this knowledge may be interpreted in an unintended way by those who do not share these intuitions and experiences. "Conversations" are needed to develop some shared interpretations, based on agreed upon definitions of key terms. Unlike Procedural knowers, Constructed knowers are concerned more about the meaning and contents of knowledge rather than about the form of knowledge, which is regarded as based on modifiable conventions.

Based on this description, the epistemological position of theoretical thinkers would be the closest to the Constructed knower. This does not imply, however, that a person in the position of Constructed knower would necessary be a theoretical thinker.

Question 7.1

This question addressed the student's attitude towards truth. We reproduce the statements representing various attitudes below, making explicit references to Belenky et al. (1997), and to the categories of epistemological positions that the statements are meant to represent. These references were removed from the version presented to the students in the interview.

Which of the following statements describes best your attitude towards truth?

A. (Silence) I am not in a position of knowing the truth or what is truth. I rely on teachers to tell me the truth or on moral authorities to tell me or, even better, show me, what's right and what's wrong.

B. (Received knowledge) Teachers are always more or less right because they have the books to look at (Belenky et al., ibid, p. 39) and they have passed all these courses before me.

C. (Subjective knowledge) I know the truth when I feel something's true, because it sort of agrees with my experience. Of course, someone else may feel otherwise; a truth for me need not be a truth for you. "I'm having hard time with the premise that truth is scientific knowledge because for me it isn't that at all. For me it's internal knowledge. I don't think we need to ascertain what's right at all. I think we need internal exploration and knowledge of self to know what's right and what is true" (Belenky et al., ibid, p. 72).

D. (Procedural knowledge) Experts have definite procedures to find the truth. You just apply the prescribed techniques, or methodologies, for example, statistical analysis to a set of data, and you draw conclusions. What you obtain is truth. There are expert procedures even for solving everyday life problems, like child rearing or managing family disputes.

E. (Constructed knowledge) Truth is something that is constructed in one's mind and shared among people, not just found in things external to the mind, although objective factors cannot be ignored. "In science you don't really want to say that something's true. You realize that you are dealing with a model. Our models are always simpler than the real world. The real world is more complex than anything we can create is. We're simplifying everything so that we can work with it, but the thing is really more complex.

When you try to describe things, you're leaving the truth because you're simplifying" (Belenky et al., ibid, p. 138).

Question 7.2

The aim of this question was to reveal the student's attitude towards the nature of mathematical knowledge. Belenky et al. (ibid.), did not say much about their subjects' attitudes toward mathematical knowledge: this was not the object of their study. We thus had to imagine or deduce what would be the attitudes towards mathematical knowledge in each of the five epistemological positions. We found it difficult to distinguish between the attitudes towards mathematics of Silent, Received and Subjective knowers and therefore we grouped these three positions together in one statement.

Which statement, according to you, describes best mathematical knowledge?

A. (Silence, Received knowledge, Subjective knowledge) Mathematical knowledge is a set of rules and formulas for the computation of exact values of some unknown quantities, given the values of some other quantities.

B. (Procedural knowledge) Mathematical knowledge is a system of tools for the modeling of the functioning of real life activities such as financial operations (banking, insurance), and for the design of engineering projects and technological devices.

C. (Constructed knowledge) Mathematical knowledge is a developing system of abstract theories about the most general aspects of relations between phenomena. Mathematical theories attempt to make their assumptions as clear as possible. Mathematical theories are made of conclusions drawn from these assumptions, rather than from some empirical evidence.

Question 7.3

This question addressed the students' attitudes towards the study of mathematics.

Which statement best describes the reasons why you are taking mathematics courses?

A. (Silence) I am taking mathematics courses because I really had no choice. I wanted to enter the profession of .... (specify) and mathematics courses are required for it.

B. (Received knowledge) I prefer math and sciences to humanities and social sciences. At least in math there are clear criteria whether you are right or wrong. "There are absolutes in math and scienceÉ[whereas] work in other courses seems to accomplish nothing. It doesn't really matter whether you are right or wrong because there isn't right or wrong" (Belenky et al., ibid, p. 42). In math you know what you are going to be evaluated on and how the grade will be computed.

C. (Subjective knowledge) I take math because I can do math, I've succeeded in it so far, so it allows me to get a university degree, but I don't think it has anything to do with my life or with truth, for that matter. I don't need math to make decisions in my life. I go by my gut feeling and not by my computations. This is why I think I haven't really learned anything in math. I have just crammed for the exams. The teachers always tell us how math is useful but then they dump on you all this theory and show some "applications" at the end, but it's all so far fetched that you would never think of using the theory if you were not told that it could be applied. I think the way I've always learned things was through experience and practice. "I like to know what's going on, so it's hard for me to explore something on the theory aspect and then go out and get the practical. I like to have the practical first, so I know what's going on, and what it's really like and then look at the theories that way" (Belenky et al., ibid, p. 202).

D. (Procedural knowledge) I am taking math and science because I trust their methods of arriving at truth. Look - the proof of math is in all those things that we, physically weak creatures, can do: fly without having wings, talk to people miles away, walk on the moon. The scientific procedures work, you just have to follow them precisely.

E. (Constructed knowledge) I am taking math because it gives you such a lot of space for imagination and creativity. You can actually do math while you are learning it. You start with a little theory and you can immediately go on and solve problems. Indeed, you can invent problems of your own and try to solve them. In social sciences or literature, you have to read pages upon pages of other people's writings and compose long essays analyzing other people's creations before (if ever) you are allowed to create something yourself.

Question 7.4

In this question we were attempting to reveal the students' attitudes towards mathematical proof. Here again, we had to imagine the possible attitudes that could be related with Belenky et al.'s epistemological positions.

Which sentence describes best your attitude towards proofs in mathematics?

A. (Silence, Received knowledge, Subjective knowledge) Proof is just a type of exercise in mathematics assignments and tests.

B. (Subjective knowledge, Constructed knowledge) Without a proof we wouldn't know if a mathematical statement is true or false.

C. (Procedural knowledge) Proofs are part of mathematical discourse and style. That's how you write mathematics. Proofs are not necessary to establish the truth of a statement; very often you are convinced just by a few examples.

D. (Constructed knowledge) Proofs are a way to establish mathematical theorems. A statement without a proof is just a conjecture.

Students' behavior in Question 7: Characterization of the group's epistemological profile

None of the students in the group belonged to the categories of Silent, Received or Subjective knowers. They had developed opinions about many epistemological and ethical questions. They were not just "quoting" received words of wisdom. They were very critical with respect to some of the statements in Question 7, and they were able to formulate their own positions as different from the proposed ones. Their opinions were developed on the basis of personal studies and readings that were not imposed on them by a system of compulsory education but were a matter of personal choices. During their college studies they sought courses outside the mathematics and science program they were all, except for one student (S2), enrolled in. Student O3, for example, took English classes. He did not agree that,

> [Reading] "Work in other [non-mathematical] courses doesn't seem to accomplish anything". No, it does. Like, English classes, you do improve yourself, you do get to write better, and even if you're going to learn mathematics and you're going to work in another field, in a mathematics field, you're always going to have to write reports, it's, it kind of, like, shows who you are, through the way you write, so, it's pretty important. And, also in these courses you develop, like, for you, a way of thinking, a way of thinking out things, you're not only seeing things through one perspective. (O3)

He and several other students took courses in the world's religions, and many took "critical thinking" courses.

None of the students appeared to fit the image of a strictly "subjective" knower, because they all exhibited some of the general features of analytic and systemic thinkers. They did not agree with the subjectivist's credo, "I don't think we need to ascertain what's right at all. I think we need internal exploration and knowledge of self to know what's right and what is true". They believed in the existence of conventional, logical, and scientific types of knowledge whose validity cannot be solved by "internal exploration". They all pointed to the differences between scientific knowledge and everyday knowledge or religious beliefs, between internal and external knowledge. They were, in general, making very subtle distinctions between concepts, which pointed to their analytical sensitivity. For example, one student distinguished the mathematics of the learner or of someone who applies mathematics in a profession from the mathematics of a teacher or of a researcher in mathematics. Another student distinguished between "fact" and "an account of a fact".

Surely, some students were more centered on themselves than others, speaking about their relationship to ideas (let's call it "subjectivist discourse") rather than about relationships between ideas ("relational discourse") (TB1c). Eight students (O3, V1, V4, S1, S2, S4, N1, N2) were consistently speaking about general relations between ideas rather than about themselves and ideas or particular examples. The discourse of four other students (O4, V2, V3, S3) was a mixture of such detached talk and a kind of confession. The two remaining students (O1 and O2) consistently kept their discourse on the "confessional" plane. However, the subjectivist perspective of the students' discourse could have been a result of their interpretation of what was expected of them in the interviews; namely, that the interviewers were interested in their personal opinions. We cannot claim, on this basis, that those who used a "subjectivist discourse" were "subjectivist knowers". Thus the only problem we had was how to distinguish between Procedural and Constructed knowers among the students. We summarize below the main differences we saw in the two epistemological positions.

Procedural knowers	**Constructed knowers**
• See themselves as users of expert knowledge	• See themselves as potential partners of experts
• Don't trust their intuitions and thus feel as outsiders to knowledge they study	• Trust their intuitions to a certain extent and thus feel they participate in the knowledge they study
• Are concerned more about form of knowledge than about its content	• Are concerned more about the content of knowledge than about its form
• See expert knowledge as unquestionable	• See knowledge as always open to questioning
• Validity of knowledge is accepted upon the word of an expert	• Feel the need to verify the validity of knowledge for themselves
• Scientific knowledge is regarded as composed of facts established beyond doubt	• Scientific knowledge is seen as hypothetical

Procedural knowers

Only two students (O1, N1) saw themselves strictly as passive "users" of a knowledge that did not belong to them ([0,1] on TB1a "Researcher's attitude"). They seemed to want to apply mathematics in their future jobs as ready-made recipes. They did not bother with the possibility that the assumptions in the mathematical models might not be satisfied in situations, to which they would want to apply them. They did not see these mathematical models as hypothetical knowledge. If they read proofs of the theorems they learned, it was only because they could contain some "facts" that could be useful in solving examination questions, not because they wanted to gain some control over the validity of a theory. We had no doubts in classifying these students as "procedural knowers". Classification of other students presented us with more difficulties.

Procedural knowers aspiring to be constructed knowers

Two students, S2 and S4, believed that the learner *should be* a potential partner of the experts by achieving some control over the meaning and validity of knowledge, and they saw mathematics not as a set of procedures but as a hierarchy of ever more abstract theories, developed by the experts, using not only rigorous and formal reasoning but also creative intuition. S4 said, "*What I think is most powerful in mathematics isÉit gives you a degree of abstraction that is higher and higher, you always go further and you always develop your ways of thinking and you always go deeper and deeper and you understand more things. And what is good, too, is that you approach a problem from many different points of view*". S4 also said, "*[apart from procedures] there is a part of intuition, a part that's not radical, you know, predefined and technical*".

However, as learners themselves, especially in linear algebra, they felt they could not live up to this ideal; they could only follow ready-made procedures, sequences of small steps, without understanding neither the meaning nor the significance of the results obtained this way. They found they could not trust their intuitions. S4 said that, in life, she could follow her intuitions, but "*not in mathematics courses, exams. This is mathematics, now we have to prove this. There is no intuition, really, in that. [I may have some expectations as to the result of a problem, and you can call that] intuition, but, the way you do the problem, there's no intuition at all. [You just follow a procedure] Yes*". These students felt that they were reduced to "studying other people's proofs" (S4), just like literature students are studying "other people's poems" without being able to write poems themselves. This made them feel miserable in the linear algebra courses, and made one of them (S2) abandon mathematics after the first term and switch to another program (Psychology). The other continued in the program and took the second linear algebra course, but found the burden of her life as a Procedural knower very hard to bear.

Here is an excerpt from the interview with S2, where her dissatisfaction with being reduced to just "doing the small steps" without understanding the general idea behind them was revealed.

> S2: [In Calculus] they had rules that seemed to make senseÉlike you are looking forÉforÉfor a square rootÉyou are looking for, you knowÉderivatives had specific rules andÉyou knew what you were [looking for]ÉLike, IÕm looking at

this function, I have to take the derivative, there are rules for that, you know what that is, IÕm doing it. ForÉfor linear, I found it was more conceptual, and the fact that I didnÕt have the concept priorÉlike maybe just a little bit tasted it, it made it very difficult, Ôcause I couldnÕt figure for the life of me what (É) I couldnÕt see what it was doing. And the fact that I couldnÕt see what it was doing made it harder.

AO: So you couldnÕt get a feel for what it was about?

S2: YeahÉbecauseÉumÉI donÕt know, the way the teacher did it, he described somethingÉand then he defined itÉwell, noÉhe gave an example, and then he defined it, but he never reallyÉ(É) But what mightÕve been better is just the description of whatÕs happÉlike, what the *whole* idea is, you know?

AS: Mhm

S2: Yeah, yeah, like, tell me what I am doing. This is what I am doing, and this isÉ thisÉ*why* I am gonna take these stepsÉand this isÉWhen I come up with an answer, my answer isÉ *this*! Like, *what does my answer mean*. ÔCauseÉthat was another problem, I got this answer but sometimes I felt it was difficult toÉ like, this is my answer but what *is* that? You know?

AS: You had no control over whether this is correct or not correct, or why would it beÉ

S2: BecauseÉyeahÉbecauseÉbecauseÉlike, for the Calculus, like, you are getting the derivative, you are gonna get a function that hasÉyou knowÉ whateverÉYou know what it is, you know what it looks like. But for the linear [algebra], I felt less confident in what I was going for, so it was also, like, *frivolous*?Énot frivolous, butÉwhatÕs the wordÉlike, it wasnÕt tangible, like, in my head. I couldnÕt say IÉ(É) [knew] where [I was] goingÉIf, you know, okay, now I am looking for this linear transformation, I know that thatÕs what I want, or I know that a kernel is this, and I get that and I look for the values associated with it. You know, if you canÉif you canÉif you can tell yourself what you are doing, I find, then you can do it, Ôcause all you have to know is the individual steps, and *every teacher teaches the individual steps, but if you donÕt know where youÕre going with it, then the steps get all confused*. You donÕt know what youÕre looking for, exactly, so the steps could be interchangeable, Ôcause you justÉ

Thus S2 was not satisfied with knowing only the sequences of "individual steps" (i.e. procedures) needed to solve the typical linear algebra exercises. She proved she could master the procedures by achieving a high grade in the first linear algebra course. But she did not feel she had control over the meaning and validity of these procedures. She could not live up to her epistemological ideals in doing and learning mathematics. Like S4, she was a *procedural knower in act, and a constructed knower in yearning*. It was characteristic of these two students that they believed in the possibility of absolutely certain knowledge. It was perhaps their lack of control over the meaning and validity of the knowledge they were learning that made them trust the experts without criticism. S4 said that "*true are things that have been* *scientifically proved or that are facts.É In science you know what is true and what is false. Two plus two is four and it's false that it makes three*". She would probably be surprised to learn that, under some assumptions, her proverbial "two plus two" could equal to zero.

These students' (S2, S4) behavior was coded [0,1] on the feature TB1a "Researcher's attitude".

Constructed knowers

Students O3, O4, V2, V3, V4 and S1 came the closest to our "canonical" description of *constructed knower* ([1,0] on TB1a "Researcher's attitude"). We describe the behavior of O3 in more detail below.

O3 saw himself as a potential partner of experts; he didn't want to go into pure mathematics and aimed at living a life of an applied mathematician, but he believed he needed to have conceptual control over the theory, too. O3 didn't say he had no difficulty in linear algebra (he said he could do it but he didn't "see it" because it was very abstract). S2 also complained about not "seeing" it in linear algebra, but, unlike her, O3 was quite confident in his abilities to achieve understanding. One way of achieving this understanding was through reading the proofs and trying to visualize the connections between the different concepts and the reasons behind the theorems. He didn't feel like a stranger in mathematics. On the contrary, he said he "*definitely wanted to include mathematics in [his] life*". However, he didn't want to participate in mathematics as a mathematician in the future, but rather as a user of mathematics. He said he is not one who would "do math while learning it" and "create mathematical problems" for himself. "*ItÕs possible that someone likes to create problems for themselves, but, not in my case, itÕs... I am not taking math courses because I can create problems and solve them*".

But, in spite of these declarations about not wanting to engage with mathematics to the point of creating and solving problems for himself, O3 felt quite responsible for the validity of the knowledge he was learning. O3 saw the need to verify the validity of knowledge by himself. He said one can rely on teachers to learn some basic things but then one must find out for oneself if and why things are true or false. He argued that one couldn't always trust one's predecessors because it would lead to a chain of lies if someone, at the start, told a lie.

> I would think that itÕs better for me to discover the truth myself. Like, instructors arenÕt there to just tell you what is true or false. If this was the truth, like where would they start,

Éwhere would they start to try to see these concepts if they all were lied on, there must be someone who had Édiscovered it. And when you discover something, whatÕs true or whatÕs false, then it stays in your mind, because you discovered it, or someone has discovered it, but you found out that it was true. ProbablyÉyou should rely on your teachers to teach you some things, and based on those you can, you can find the truth about other things. Like they show you some building blocks, and you use the building blocks to build other things, but theyÕre not just here to show you everything. (O3)

He considered that it is not enough to just learn the theorems. You need to know their proofs, so that you know why you are accepting their truth. He said he didn't particularly like studying the proofs but he considered it useful. "*I donÕt always, like, remember everything in the proof, but I do get a better general idea of whatÕs happening*". He hated doing "proof exercises", yet would do them:

Yeah, I know theyÕre important, but I hate them because theyÕre a big hassle. You really have to work a lot of time on it. ItÕs not like a specific example you have to do. When you have so many things to do, and you have a proof, you canÕt say itÕs going to take me 5 to 15 minutes, because for that problem probably itÕs going to take you much more. Or, youÕll have to seek some help. But letÕs say for an example, theyÕre all similar. So, you just have to follow the technique and practice on it. (O3)

O3 viewed scientific knowledge as providing approximate models of reality rather than some kind of absolute truth. "*Experts have ways to find, I wouldnÕt say the truth, but it would be more a portrayal of life. LetÕs say for the statistical analysis, I donÕt see a relationship with truth. Really, itÕs, itÕs more, itÕs more a portrayal of, of how, letÕs say the society is, or howÉItÕs like a model. ItÕs not necessarily the truth. But, experts do have procedures to find these things*". However, one must be careful in applying these procedures for concrete problems because "*they may not be efficient for that specific problem*".

O3 attributed an important role to intuition in learning mathematics. Understanding, for O3, appeared to consist in "seeing" things. He said he sometimes didn't "see it" in linear algebra, the way he could see things in calculus, literally visualizing functions as curves and surfaces. In linear algebra, the images were not as visual, but he appeared to visualize abstract configurations of concepts:

How do I read [proofs]? I start by reading the theorem. What theyÕre claiming. And try to see in my mind, try to see a picture of what are they talking about, like this part is that part, so what is this and what is that, and try to see a connection, or looking at properties, and trying to find a way to get from one to the other, or. [I make] a lot of notes, and drawingsÉIt helps because the vision is like the faculty of learning. (O3)

O3 was concerned more with the content than with the procedural aspects of mathematics, although he admitted that the importance of the latter should not be underestimated. But he rejected the description that "Mathematical knowledge is a set of rules and formulas for the computation of exact values of some unknown quantities, given the values of some other quantities". He said that this is perhaps a portrait of high school mathematics, but not university mathematics. A formula is useless if you don’t know what it means:

> [The formulas] are useless if youÉanyone can just put in numbers and calculate the results, but what does that mean? The meaning is very important. YouÕre not alwaysÉ going to use a formula and plug in numbers and get results, you really have to know when to use it, how to use it, why is it used, and where does it come from? How would that person think about it to start building on? (O3)

O3 saw knowledge not as something one achieves at school and then only uses it in a profession, but as a lifelong endeavor. One learns all one’s life. Experts have limited knowledge just like everybody else. "*I canÕt expect [teachers] to know everything, either, theyÕre also human even though theyÕre teachers. So theyÕre not always right, but they know what they know*". So knowledge can be questioned, but O3 found mathematics less questionable than human science, because, he said, mathematical knowledge is more objective. He believed that there are right answers and wrong answers and criteria of evaluation don’t change from teacher to teacher, as it often happens in human science classes.

Students whom we classified above as "constructed knowers" all insisted on the role of intuition in mathematics and the need to personalize knowledge in learning it. This point was especially stressed by O4 and V4. O4 referred to the case of his cousin whom he was helping with mathematics and who had a lot of difficulty with it. O4 attributed her difficulties to not using her intuition: "*[math] is sort of not in her*". Using intuition was linked, in V4, with rejecting the proceduralist attitude. Intuition meant relying on one’s own judgment and becoming an expert oneself. He said that "*there’s expert knowledge but there is my own judgment, too, so the expert is my expert. You take into account what the expert says but then you apply it to youÉThere is a level of application of what the expert does. If some people just follow it, it’s because maybe they cannot deal without it, they don’t have the power toÉ*". It is by transcending procedures that new knowledge can be found: "*If Einstein just followed the rules, he would have followed Newton’s laws and wouldn’t have discovered much*". Mathematics, for him, was "*just one tool among others*". Mathematical models could give "a clear indication" in making some decision, but they were not to be regarded as right or wrong. He referred to financial applications of mathematics and pointed to the limited applicability of mathematical models in this area: "*Say, a company wants to make an investment and,* *according to numbers, it is good, butÉSay, the stock market, it doesn't go up because of good numbers but because there is news*". Although both O4 and V4 believed in the existence of some absolute truth, both quoting the common "wisdom" that "*truth is there but you have to find it, which can be difficult*", they considered all existing scientific knowledge as built of tentative models, based on axioms or assumptions. Their notion of the hypothetical character of knowledge was not very clear, however. For example, V4 stressed not so much the dependence of all scientific claims on assumptions that may or may not be satisfied in a given context of application, but the fact that scientific theories are not universally "right". He was not stressing that scientific theories are just theories, but that scientific theories are not true descriptions of reality. He said: "*science it's models that, so far, haven't been disprovedÉ Like Newton's laws, they happened to be wrong at a certain point*".

Constructed knower of technical mathematics

Student O2 was a difficult case. We were tempted to classify him as a Procedural knower, because he appeared to be interested only in mathematical techniques and didn't see any purpose in studying theorems and proofs. But we would thus be making the mistake of confusing the knowledge the student was learning and his or her epistemological position with respect to this knowledge. And when we focused on his epistemological position with respect to the technical knowledge he wanted to learn in mathematics, then we became convinced that he had a Constructed knower's attitude rather than the Procedural knower's attitude.

He appeared to consider himself a potential *expert user of expert knowledge*. He did not feel the need to have any kind of control over the theory; this was the expert's business and duty. But he wanted to be comfortable and masterful in solving problems using this theory. He believed he could develop enough experience to achieve control over the methods of solving problems (in financial mathematics). For now, the way he controlled his solutions was whether "*it comes out right answer"*. He would not "*use a proof to show this is true or false"*. He would indeed be convinced by a few examples. He thought that proofs were necessary only for teachers; this was the teachers' part of knowledge. The students' part was to be able to solve problems; therefore, understanding examples was enough. He didn't think he would have to have recourse to proving even as an autonomous worker in an insurance company, upon encountering some contradiction. All he would need is to know how to solve the problem, and for this all he would need would be practice and experience, and method, not a proof within a theory. Within this frame of mind, the role of the notion of "hypothetical character of scientific knowledge" was played by the notion of "tentative character of all knowledge based on experience".

> I say practice and experience in real life is more important than... the theory. I'm not saying in math to prove is not real, but all proof in others…in accounting, in finance, I don't think there to prove is necessary. Only to...let... let's... trust us. To make us like, 'Oh, the book is right, because they have a proof. They know it is right'. But just to make me like, more confidence, like, 'Oh, it is right, this proof is right. So, if I use this method, this method is right, too'. But what I understand, what I know, is only the method, not the proof. (…)
>
> Usually, theory in class, I didn't listen well what the teacher say…it is, 'Oh yeah, yeah, yeah, yeah, yeah'. You know, sometimes I don't even write it down, for theory or proof. I think it's no use. But for problems, when they do some problems, example, I really write it down, I have to listen. So, so it's not really the theory make me understand this, but to solve problems, the example, make me understand and do the problems. Even in the book, they give me a lot of proof, a lot of theory, what... what's this? I don't even understand. But they... if they give me an example, I can use the example to understand what is going on, or I also [learn] which method I have to use. It's better than theory. (O2)

For O2, mathematics was about solving problems, and problems were a way to understanding mathematics. He wanted to learn mathematics in an active way, and following someone else's reasoning by reading a proof was boring and unproductive. He was interested in solving problems because it was the only way for him to make knowledge his own and have some control over it. He refused being interested in theory but his conjecture in the "log-log" question suggests that he was capable of posing theoretical problems as well. Perhaps he was not sufficiently interested in the theory of linear algebra to engage in posing theoretical questions in this domain. Thus, with respect to the domain of linear algebra, we decided to classify his position as that of a *Constructed knower of technical mathematical knowledge*. His behavior was coded [1,1] on the feature TB1a 'Researcher's attitude".

Constructed knowers of applied mathematics

Three other students (V1, S3, N2), like O2, also appeared to aspire to become *expert users of expert knowledge*, but, unlike O2, they considered it necessary to know and understand the theory (including proofs of theorems) in order to achieve this expertise in applications. This was perhaps worth the effort, because they passed the second linear algebra course (S3 and N2 with A's, and V1 with a B), while O2 failed it. We classified these students as "Constructed knowers of applied mathematics", coding their behavior as [1,1] on the TB1a feature.

V1 considered mathematics as a system of tools for solving both large scale and small-scale real life problems, and said that to the extent that it has such applications "*it has a* *lot to do with my life and truth*". He agreed that "*we need the proofs to establish if the theorems are true or not*" but added that "*I am not the one who can write them, and, even if they are interesting, I find the applications more useful to learn*". He said that he would normally start by reading the theory and making a short summary of it, then he would go on to read the applications and do the problems and eventually, he would go back to the theory and read it in more depth for a better understanding. He did not claim control over the validity of the proofs ("*Sometimes proofs are very complicated so we cannot always check [them]"),* but he did not blindly accept the theory as valid. His notion of validity of mathematical theories was not focused on their internal coherence but on their applicability. He considered mathematical theories as having only limited applicability to real life situations, as approximate models. He mentioned statistical models as based on an idea of the "*average person*", but real people could be very far from this "*average person*". For him, "true" did not really apply to scientific knowledge but he said, "*Most of it makes sense*". It is up to the user of these models to verify if they apply in a given situation — this was the part of the responsibility for the validity of knowledge that he would assume.

Understanding of the hypothetical character of scientific knowledge was quite explicit in three students (O3, V3, S1; [1,0] on TB23a). Other students were not clear about this issue, but several expressed skepticism with respect to the scope of the validity of scientific knowledge. Five of them (O4, V1, V4, S3, N2) mentioned the limited or "approximate" character of mathematical models with respect to the reality they are supposed to represent (we coded this TB23a behavior as [1,1]). In the case of O2, "hypothetical" meant something close to "tentative" and applied to technical knowledge grown from experience (behavior likewise coded [1,1]). Student V2 accepted the conditional character of all knowledge except for logic, which he believed to be one and universal for all human thinking ([1,1]). All students classified as procedural knowers appeared not to be aware of the hypothetical character of scientific knowledge; they were not posing themselves this kind of questions (O1, S2, S4, N1). It was not up to them to judge of the nature of the validity of scientific knowledge; they did not participate in its construction.

If we admit that students whose behavior was coded [1,0] on relational thinking, researcher's attitude and hypothetical thinking qualified as Constructed knowers then there were two clearly Constructed knowers in the group: O3 and S1.

Question 7. "Beliefs"						
	TB1c (relational)		TB1a (researcher's)		TB23a (hypothetical)	
O1	0	1	0	1	0	1
O2	0	1	1	1	1	1
O3	1	0	1	0	1	0
O4	1	1	1	0	1	1
V1	1	0	1	1	1	1
V2	1	1	1	0	1	1
V3	1	1	1	0	1	0
V4	1	0	1	0	1	1
S1	1	0	1	0	1	0
S2	1	0	0	1	0	1
S3	1	1	1	1	1	1
S4	1	0	0	1	0	1
N1	1	0	0	1	0	1
N2	1	0	1	1	1	1

Table 7. Students' behavior in Question 7 "Beliefs". Only two students behaved consistently theoretically in this question.

CHAPTER III. RELEVANCE OF THEORETICAL THINKING FOR SOLVING FINAL EXAMINATION QUESTIONS

Our question in this chapter is: how relevant, *in principle*, was theoretical thinking for the solution of the final examination questions in the two linear algebra courses taken by the interviewed students? We start by giving some information about the content of the courses, and then we proceed to analyzing the two sets of examination questions.

THE CONTENT OF THE LINEAR ALGEBRA COURSES TAKEN BY THE INTERVIEWED STUDENTS

The linear algebra courses for Actuarial, Mathematics specialization and honors students at our university are focused on theory and proofs. Applications are not tackled in these courses. Students see applications of linear algebra in other courses (for example, in "Operational research") later on in their studies.

In the sequel we shall use abbreviations "LAI" and "LAII", in reference to the first and second linear algebra courses taken by the interviewed students.

Content of LAI

The textbook used in the course was: Johnson, L.W., Riess, D.R., Arnold, J.T.: 1993, *Introduction to Linear Algebra*. Addison-Wesley. The contents of the course could be summarized as follows: Systems of equations; Matrices: operations on matrices; Vector spaces R^n; General vector spaces; Basis, dimension, coordinate vector; Eigen-theory for matrices (up to diagonalization); Linear transformations; Eigen-theory for linear operators.

The syllabus of the course included approximately 34 definitions and 29 theorems or computational procedures (such as the Gaussian elimination method or computation of the inverse of a matrix); in total, 63 "units" of theory. One can evaluate the average time needed to "cover" such a unit of material in the lectures, as follows. Given that the course spanned over 13 weeks, with 2.5 hours of class time per week, and counting away two weeks (for revisions and the class test), the lecture time could be estimated at 27.5 hours or 1650 minutes. This gives an average of 26 minutes of lecture time per unit.

Content of LAII

Chapters 8, 9, and 10 of Gilbert, J. & Gilbert, L. 1994, *Linear Algebra and Matrix Theory*, Academic Press, were used as a text for the second course. The syllabus included the following topics: Functions of vectors: Linear Functionals, Real quadratic forms, Orthogonal matrices, Reduction of real quadratic forms, Classification of real quadratic forms; Inner product spaces (over **R**): Inner Products, Norms and distances, Orthonormal bases, Orthogonal complements, Isometries; Spectral decompositions: Structure of linear operators on finite vector spaces, including the Cayley-Hamilton theorem, Spectral decomposition theorem and Jordan Canonical form theorem. Altogether, 35 definitions and 48 theorems from these chapters were included in the syllabus of the course. In total - 83 units of theory. The average lecture time per such unit was thus 1650/83 or about 20 minutes.

ANALYSIS OF THE FINAL EXAMINATION QUESTIONS IN LAI

We start by reproducing the examination questions.

The examination questions

Question LAI.1a (6 marks)

Let a be a fixed real number. For which values of a is the set
{(a, -1, 5), (3, -2, -a)} linearly dependent?

Question LAI 1b (6 marks)

Let A, B and C be 3x3 matrices and let B and C be elementary matrices such that

ABC = $\begin{bmatrix} 1 & 4 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. What is the row reduced echelon form of A^T and how many operations are needed to obtain this reduction? Explain.

Question LAI.2a (8 marks)

Find a basis of Span(A) if
A = {(-1, 4, 5, 1), (1, -1, 1, 2), (1, 0, 3, 3), (0, 1, 2, 1)}

Question LAI.2b (7 marks)

Let A= {(1, 1, -1), (0, 4, 3)} and B = {(2, 1, 2), (3, 2, 4), (1, 0, 1), (2, 6, 3)}
Determine whether or not Span(A) = Span(B). Explain.

Question LAI.3a (6 marks)

Let A = {(2, 1, 0), (1, -1, 3), (1, 0, 1)}

Let A = $\begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & -1 \\ 1 & 3 & 2 \end{bmatrix}$

Find B so that A is the matrix of transition from A to B.

Question LAI.3b (6 marks)

Let A = $\begin{bmatrix} 1 & 2 & 1 & -1 \\ 0 & 1 & -1 & 2 \end{bmatrix}$ be the transition matrix from A = {(0, 1), (1, 1)} to the set B.

Let B = $\begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 1 & 0 \\ -2 & 1 \end{bmatrix}$ be the transition matrix from B to C . Find the transition matrix from A to C

and find the set C.

Question LAI.4a (7 marks)

Give two 2x2 matrices that are column equivalent but not row equivalent.

Question LAI.4b (8 marks)

If A = $\begin{bmatrix} 1 & 5 \\ 2 & 10 \\ 2 & 9 \end{bmatrix}$ and B = $\begin{bmatrix} 2 & 3 \\ 4 & 6 \\ 9 & 13 \end{bmatrix}$, find an invertible matrix P so that B = PA and find an invertible matrix Q so that $B^T = A^T Q$.

Question LAI. 5 (6 marks)

Find a basis for the space V of 3x3 symmetric real matrices. Give another vector space isomorphic to V.

Question LAI.6a (7 marks)

Let T be the mapping of R_{2x2} into R_{2x2} as defined by

$$T\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) = \begin{bmatrix} 2a_{21} - a_{22} & 2a_{11} + a_{12} \\ a_{21} - 3a_{12} & a_{11} + 3a_{22} \end{bmatrix}$$

Show that T is a linear transformation. Find a basis for the kernel of T and a basis for the range of T. Find the rank and the nullity of T.

Question LAI.6b (8 marks)

Let $u_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $u_2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, $u_3 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $u_4 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

Given that $B = \{u_1, u_2, u_3, u_4\}$ is a basis of R_{2x2}, find the standard basis of Span (A) relative to B, where $A = \{A_1, A_2, A_3, A_4\}$ and

$A_1 = \begin{bmatrix} 2 & 2 \\ 2 & 1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$, $A_3 = \begin{bmatrix} 6 & 7 \\ 5 & 3 \end{bmatrix}$, $A_4 = \begin{bmatrix} 4 & 7 \\ 1 & 2 \end{bmatrix}$

Question LAI.7 (7 marks)

Let T be a linear transformation of R^3 into R^2 whose matrix is

$A = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 4 & -2 \end{bmatrix}$ relative to the standard bases E_3 and E_2. Find the matrix of T relative to the basis {(1, 1, 1), (1, 2, 0), (1, 1, 0)} of R^3 and {(1, 0), (1, 1)} of R^2.

Question LAI.8a (10 marks)

Let $T(a_o + a_1 x + a_2 x^2) = (3a_o + a_1 + a_2) + 2a_1 x + 2 a_2 x^2$ where T is a linear operator on P_2. Find the eigenvalues of T. For each eigenvalue, find the algebraic multiplicity, the geometric multiplicity and find a basis of each eigenspace. Can T be represented by a diagonal matrix? If yes, find a basis of P_2 such that T is represented by a diagonal matrix relative to that basis.

QuestionLAI.8b (8 marks)

Prove or disprove: matrices of the same rank are similar.

We now proceed to analyzing the examination question in LAI.

Relevance of theoretical thinking in the solution of LAI examination questions

We consider each feature of our model of theoretical thinking in turn and ask, to what extent this particular feature was relevant in solving the questions.

Relevance of reflective thinking

In general, it is not in the interest of the student, during an examination, to engage in "thinking for the sake of thinking", getting absorbed in mathematical investigations, if this activity is not directly related to producing acceptable answers to the examination questions, and is not rewarded by marks. One could go as far as saying that even if the examination questions did require mathematical investigations, the student would not be thinking about the questions for the purpose of gaining new knowledge or a better understanding of some idea but for the sake of getting marks. Thinking for the sake of thinking is simply contradictory with the institution of limited-time final examinations. However, in the same vein, one could claim that a mathematician doing his or her research is likewise not thinking for the sake of thinking but for the sake of, say, gaining fame and recognition or, more modestly, simply writing more publications and thus increasing his or her chances of getting a promotion or a funding. This may be true, but there is nevertheless a time in this mathematician's thinking where the mundane goals of his or her activity are overshadowed by an intellectual excitement. With this in mind, it makes sense to ask, perhaps, if the examination questions gave the students a chance to temporarily forget about the examination situation and engage in solving an intellectual puzzle.

To answer this question one would naturally look for open-ended problems, or at least problems allowing for several possible solutions in the test. But there were no open-ended questions in the LA I examination paper. Three questions did not have a unique answer (LAI.2a, 4a and 5). These questions required examples of objects of a certain kind: a basis of a given vector space, two 2x2 column equivalent matrices which are not row equivalent, a vector space isomorphic to the vector space of 3x3 matrices. While infinitely many examples exist in each case, a well-prepared student would be familiar with some standard examples given in the course and the solution of the three questions would not require much in the way of "mathematical investigation".

Another aspect that could be considered in deciding whether an examination paper required reflective thinking or not would be the predictability of the questions used: how different were the questions from those used in the past examinations? Even a "closed" question with a unique answer could present a major intellectual challenge if it was a novel question the student had never seen before. This particular LAI examination paper was very similar to its predecessors given by the same instructor, and past tests were available from the university copy center. The only unpredictable questions were questions LAI.4a (example of column but not row equivalent 2x2 matrices, 7 marks) and 8b (true or false: same rank implies similarity of matrices, 8 marks). This is an important factor to consider in explaining why some students, not necessarily strongly inclined to theoretical thinking, were able to achieve high grades in this course. Indeed, student S2 explicitly attributed her high grade in the first linear algebra course to the fact that the final examination questions were predictable.

We conclude that reflective thinking appeared relevant in 2 out of the 14 questions (LAI.4a, 8b).

Relevance of systemic thinking

Systemic thinking in general, in the sense of thinking based on relations between concepts and not on relations between words and things or between the things themselves, was unavoidable in solving the examination questions. In all questions the solution required making appropriate connections between concepts mentioned in the formulation of the questions and their other characterizations or properties. But these connections could be based on procedures and memorized examples as well as on deduction and other kinds of more formal reasoning.

We shall consider the "definitional" and "proving" features of theoretical thinking together because they are often inseparable in mathematical reasoning (proofs require reference to formal definitions and properties).

Relevance of definitional thinking and of the activity of proving

We look at each question in turn from the point of view of the importance of definitional thinking and proving activity in providing a solution.

At first sight, Question LAI.1a may appear as requiring substantial systemic thinking: it was not enough to have a "feel" for linear dependence of vectors to solve the problem. The students were not asked to recognize ("by inspection") if a given set of vectors was dependent but were required to find conditions under which a certain set of vectors is dependent. However, it was not necessary to even know the general definition of linear dependence of a set of vectors. There were two vectors in the set and it was enough to use the characterization of linear dependence of two-element sets by the condition that one of them is a multiple of the other[7]. The question could then be interpreted as: is there a scalar k such that (a, -1, 5) = k(3, -2, -a)? But even this reasoning was not necessary. A correct solution could be obtained by associating "linear dependence" with the procedure of row reduction of matrices and applying the "test": "it's dependent if the matrix reduces to a matrix with a row of zeros" (where the reference of "it" is often quite vague in the student's mind). If the student recalled the principle "it is not allowed to multiply a row by a zero", he or she would be led to considering two cases: a = 0 and a › 0 and row reducing the matrix composed of the two given vectors as rows. Justification was not required in the problem, so students' thinking could go only as far as devising a strategic plan for the solution, stopping short of a theoretical justification of the plan. Thus, in fact, maximum marks could be achieved in this question with no definitional thinking and no concern for systemic validation.

Question LAI.1b could not be reduced to a computational exercise: it was not sufficient to just reduce a matrix to a row reduced echelon form (rref), because this matrix was not given in an explicit numerical form. But a definition of the rref was not necessary; it was enough to recognize a rref when seeing one and being able to tell a rref from a matrix not in rref. But the student had to know the theorem that the rref of a matrix is obtained through operations equivalent to multiplication, on the left, by elementary matrices. The definition of elementary matrix was not necessary, but being able to represent an elementary operation by an elementary matrix was. One also needed to know certain formal properties of transpose and not only how to transpose a matrix. Equipped this way, the student could obtain the relation: $C^T B^T A^T$ = [[1,0,0],[4,0,0],[0,0,0]][8], understanding that C^T and B^T were still elementary matrices and that the matrix on the right was, therefore, a result of two steps of row reduction of the matrix A^T. Then, recognizing that the matrix on the right is not yet in a rref, and reducing it in one step, the student could get to the conclusion that three steps are necessary for the row reduction of A^T.

In this question the students were explicitly asked to "explain". Students who only described how they obtained the answer, would not be granted full marks. For some students an "explanation" would be a description of what they did, or how they got their answer, e.g. "I transposed the matrix [[1,4,0],[0,0,0],[0,0,0]] and then row reduced the transpose in one step

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[7] Some students generalize this characterization to arbitrary dimension and believe, ignoring the definition, that a set of vectors is linearly dependent iff all vectors in the set are multiples of a single one.

[8] For the sake of saving space we write the matrix as a list of its rows.

to [[1,0,0],[0,0,0],[0,0,0]]; this is the rref of A^T; C^T and B^T are two steps of reduction, so altogether 3 steps were needed to get to the rref". This, however, would not be granted full marks; the instructor would normally expect the students to explain *why* their answer was valid, not *how* it was obtained. Explanation "why" rather than "how" would require making explicit some conceptual connections, and reference to theorems such as "an elementary operation on a matrix is equivalent to multiplying the matrix on the left by a corresponding elementary matrix"; "the transpose of an elementary matrix is an elementary matrix"; "the transpose of a product of matrices is the product of the transposes of the matrices taken in reverse order". Thus, although it was not necessary to refer to definitions in solving the problem, reference to formal properties of the objects involved in it was, and it was necessary to combine several of these properties in a chain of deductions.

In Question LAI.2a it was not necessary to refer to definitions because the question did not require any explanations or justifications. The problem was a typical exercise in the course. Some students might have interpreted the expression "finding the basis of a span" as a signal to automatically put the vectors in a matrix and row reduce the matrix. A common mistake of students who operate this way is putting the vectors into the columns of the matrix. The question could also be solved with some understanding but "by inspection" and without reference to formal definitions of "span" and "basis". The student notices that the first two vectors are redundant because they can be easily obtained as combinations of the last two: $v_1 = -v_3 + 4v_4$ and $v_2 = v_3 - v_4$. The last two vectors in the set are "obviously" independent so these two are proposed as a basis. Of course, one could produce a formal argument but this argument was not required for obtaining full marks on the question.

Question LAI.2b could not be solved using only a mechanical procedure. It was not possible to just perform a row reduction and leave the result without interpretation. Moreover, this question asked for an explanation. It was rather obvious from the formulation of the question that the students were expected to explain not how they arrived at an answer (description of a strategy) but why they considered the spans to be equal or not equal. In general, comparing two spans may require quite substantial reasoning. For example, if the dimensions of the spans were equal and their given or computed bases were not identical then it would be necessary to refer to the definition of span to check if the two sets were equal. But, in the particular case of the question LAI.2b, it was easily found that the dimensions of the spans were not equal, which immediately solved the problem in the negative. However, in this particular question students could arrive at the right answer for all kinds of wrong reasons. An acceptable approach to the problem would be to row reduce matrices [[1,1,-1],[0,4,3]] and [[2,1,2], [3,2,4], [1,0,1], [2,6,4]] to, say, [[1,1,-1], [0,4,3]] (it is already in a ref) and [[1,0,0], [0,1,0], [0,0,1], [0,0,0]], respectively, notice that their ranks are different and conclude that the spans are different because they have different dimensions. But the student could use the approach thinking thinking that the spans are different because they are composed of different (finite) numbers of vectors. Some students understand span({v_1, …, v_n}) as the set of non-zero rows of a matrix obtained from the matrix [v_1 , …, v_n] after row reduction, and dimension - as the "number of vectors in a space". But if the student did not write anything revealing his or her flawed understanding of span or dimension, full marks would be awarded for the solution. We conclude that the necessity of proving and reference to definitions or other characterizations and properties was rather weak in this problem.

Question LAI.3a could not be solved without reference to the definition of the matrix of transition from one set of vectors of a vector space to another set of vectors in the same vector space: If the set A has n elements and the set B has m elements, then the matrix of transition is an n x m matrix whose j'th column is composed of the coefficients in the representation of the j'th vector in B as a linear combination of the vectors of A. However, the problem is only a straightforward application of the definition: since the matrix A is of order 3, then the set B must have 3 vectors, say, B = {v_1, v_2, v_3}and, by reading the columns of the matrix A,

v_1 = (2, 1, 0) — (1, -1, 3) + (1, 0, 1) = (2, 2, 2)

v_2 = 2(2, 1, 0) + 0(1, -1, 3) + 3(1, 0, 1) = (7, 2, 3)

v_3 = (2, 1, 0) — (1, -1, 3) + 2(1, 0, 1) = (3, 2, -1).

Very little in the way of proving was necessary in this question. The reference to a definition could be understood as following a procedure. But at least the student had to refer to the definition.

In Question LAI.3b the student also had to refer to the definition of the transition matrix from one set of vectors to another. It was enough to apply the definition in the same way as in the previous question, twice, to solve the problem. The vectors of C could be computed by first reading their representations as combination of vectors from B and then reading the vectors from B as combinations of vectors from A , as follows: Since A is 2x4 then the set B has 4 vectors. Since B is 4x2 then the set C has 2 vectors.

Denote by $v_1 = (0, 1)$ and $v_2 = (1, 1)$ the elements of A.

Denote by w_1 , w_2 , w_3 , w_4 the elements of B and by u_1 , u_2 the elements of C.

By definition of transition matrix,

$u_1 = w_1 + 2w_2 + w_3 — 2w_4$, $u_2 = 3w_1 + w_2 + w_4$

$w_1 = v_1$, $w_2 = 2v_1 + v_2$, $w_3 = v_1 — v_2$, $w_4 = -v_1 + 2v_2$.

Hence

$u_1 = v_1 + 2(2v_1 + v_2) + v_1 — v_2 — 2(-v_1 + 2v_2) = 8 v_1 - 3v_2. = (-3, 5)$

$u_2 = 3v_1 + 2v_1 + v_2 -v_1 + 2v_2 = 4 v_1 + 3v_2 = (3, 7)$.

Of course, this lengthy method could be avoided if the student remembered the theorem that if A is a transition matrix from set X to set Y and B is a transition matrix from set Y to set Z, then AB is the transition matrix from X to Z. Then the problem is solved by applying the definition of the transition matrix only once (to find the set C). In any case, reference to theory is needed, but no substantial proving activity is necessary.

In Question LAI.4a, it was not necessary to know the definitions of row and column equivalence of matrices; it was enough to know how to column and row reduce matrices and understand row equivalence procedurally as the possibility to row reduce to the same matrix. One could start the problem by taking a simple 2x2 matrix, say, A = [[1,2],[0,0]] and column reduce it to B = [[1,0],[0,0]]. This matrix is obviously not row equivalent with A, because they are in rrefs which are different. The question did not require any explanation or justification and the example was not difficult to find. It would be a little harder to find an example of two distinct matrices that would be both row and column equivalent. This could lead to looking for sufficient conditions for two 2x2 matrices to be both row and column equivalent, which is an activity engaging substantial definitional thinking and proving activity. Another missed opportunity in this question was the student's ability to disprove a general statement by deriving a contradiction. The question could be asked indirectly, as follows: "Are column equivalent matrices necessarily row equivalent? Justify." This kind of question could even engage hypothetical thinking, because the student would have to consider the possible consequences of a positive and a negative answer.

Question LAI.4b was computationally quite complicated; it was not easy to simply guess what row (or column) operations to perform to obtain B from A and deduce the matrix P by representing these row operations as elementary matrices. Trying to find the matrix P in a straightforward computational way, by solving a system of equations leads to a system of 6 equations with 9 unknowns (the entries of P), with the condition that det(P) › 0. The solution results in 3 free variables:

$$P = \begin{bmatrix} -12-2t_3 & t_3 & 7 \\ -24-2t_2 & t_2 & 14 \\ -55-2t_1 & t_1 & 32 \end{bmatrix} \qquad \det(P) = t_2 - 28 t_1 t_3 - 2t_3 + 14 t_1 t_2 \text{ › } 0.$$

Taking, for example, $t_1 = t_3 = 0$ and $t_2 = 1$, one gets det(P) = 1 and an invertible matrix P = [[-12, 0, 7], [-26, 1, 14, -55, 0, 32]] which solves the first part of the problem. The second part of the problem could be solved in the same straightforward computational way. This approach would not engage much in the way of definitional thinking or proving. But it was quite ineffective, given the ample opportunities of making computational mistakes in solving the equations by hand.

Reference to theory and reasoning could facilitate the task enormously. For example, the student could notice that A and B are row equivalent, since they are row reducible to the same matrix [[1,0],[0,1],[0,0]]. Keeping track of the row operations and representing them by elementary matrices, K A = M B is obtained, where K and M are the products of these elementary matrices used for reduction. Since M is invertible, $B = M^{-1} K A$. It is then enough to take $P = M^{-1} K$, which is invertible as a product of invertible matrices. The computation leads to K = [[-9,0,5], [2,0,-1], [-1,1/2,0]] and M = [[-13,0,3],[9,0,-2],[-2,1,0]]. Therefore P = [[-12,0,7], [-25,1/2,14],[-55,0,32]]. By the properties of the transpose, $B^T = (PA)^T = A^T P^T$, and therefore $Q = P^T$.

It appears rather unlikely that, without a computational device capable of handling systems of equations, a student who chose the straightforward computational way, without any reference to theory and proving, would arrive at a correct solution within the time frame of the examination. Thus, we assumed that, whoever obtained a correct answer to the problem, arrived at it using theoretical means.

The above discussion has yet another point. In linear algebra courses students are often not given a chance to even try the straightfoward computational approach. There would be no time wasted on this, with barely 26 minutes of lecture time per syllabus unit! But if time was allowed for this activity, with students left on their own and getting into messy computations, the necessary *practical* experience would be built *against* which the theoretical approach could stand a better chance of being appreciated by the students.

Question LAI.5 required only an intuitive notion of basis and isomorphism; no reference to definitions was needed, since the question did not ask for any justification of the answer. The subspace of 3x3 symmetric matrices is often used as a paradigmatic example of a subspace in the lectures, and its standard basis is normally written on the board. It was thus enough to recall this example. Isomorphism needed to be understood only as "same dimension then spaces isomorphic" and "dimension n then space isomorphic to R^n". Again, memory of the paradigmatic examples would be enough to answer the question correctly. The notion of isomorphism, introduced in the course, was an opportunity to develop the students' axiomatic reasoning. Had this reasoning been valued in the course, a different question about isomorphism could have been asked on the final examination. For example, "Prove that any n-dimensional vector space over **R** is isomorphic to R^n"; or, "Prove that isomorphism preserves linear independence of sets of vectors".

Proving the linearity of the given transformation in Question LAI.6a could be approached by verifying the conditions of the definition. The student could be viewing the conditions of the definition as axioms as well as steps of a procedure, and there would be no way of distinguishing between the two ways of understanding. In any case, there is not much in the way of proving activity in the verification of the definition conditions approach.

Less tedious approaches were possible. For example, the student could use the fact that there is an isomorphism between the space of 2x2 matrices and R^4, and represent T by a matrix transformation S from R^4 to R^4: S(X) = AX where A = [[0,0,2,-1],[2,1,0,0],[0,-3,1,0],[1,0,0,3]]. The first part of the question could then be solved by reference to the fact that matrix transformations are linear. Representation of matrix spaces and polynomial spaces as R^n spaces is a standard procedure in linear algebra courses, and students would be more likely to choose this route than the approach by definition.

To answer the questions about the bases of the kernel, and the range of T, the student could apply the definitions of the kernel and range in a straightforward fashion. While the definition of the kernel leads directly to solving a homogeneous system and requires no complicated reasoning, applying the definition of range to find a general form of its elements can be more difficult, especially, if the context is not that of the R^n spaces. If T: V --> W is a mapping then range(T) is defined as the set of all vectors T(v) where v is in V. A student able to find a basis of the range from this definition would be also able to bypass this tedious route and use more powerful characterizations of range to deal with the problem. On the other hand, these more powerful characterizations of range (such as, in finite dimensional cases, the column space of a matrix of the transformation) are usually learned as "tricks" rather than conceptual shortcuts. To find a basis of the range, the student simply column reduces the matrix of the transformation.

Indeed, in solving this problem no formal understanding of basis and dimension was needed. Intuitive or even procedural notions were sufficient (dimension being obtained by counting the vectors in a basis, obtained by row reducing a matrix). In the R^4 model, ker (S) = nullspace (A), range (S) = column space (A), nullity (S) = nullity (A), rank (S) = rank (A). In the case of the given problem, A reduces to an identity matrix, therefore nullspace (A) = {0}. Hence ker (S) = {0} where 0 is the vector (0, 0, 0, 0) and ker (T) = {0}, where 0 is the 2x2 zero matrix, and there is no basis. Now, the case of no basis is not easily understood by students who rely only on "tricks" in solving this type of problems. Often these students end up saying that {0} is a basis and that the dimension is 1. If they use the equality nullity (A) + rank (A) = 4, these students conclude that the rank of the transformation must be 3, and their solution obtains very low marks.

It thus seems that, while explicit reference to definitions and theorems in writing a solution to this problem was not required, some definitional thinking and reasoning had to be used to arrive at correct solutions, due to the extreme case of the zero kernel.

Question LAI.6b could be considered as a computational exercise. It is not atypical, and has appeared on previous final exams. A student with good memory could perhaps train him or herself in following a procedure, but the procedure is long and one may easily get confused if one has no conceptual control over the steps. Here is a possible procedure: 1) represent the matrices in the set A as R^4 vectors, put them into rows of a matrix and row reduce the matrix; 2) represent the rows of the reduced matrix back as 2x2 matrices; you obtain a basis of Span(A); 3) represent the matrices of this basis as linear combinations of the vectors of basis B; you get the coordinate vectors of the basis of Span(A) relative to the basis B; 4) put these coordinate vectors as rows of a matrix and row reduce the matrix; you obtain the coordinate vectors relative to B of a standard basis of Span(A) relative to B; 5) represent these coordinate vectors back as matrices. Following this procedure one obtains a basis of Span (A), as the set of the matrices E_1 = [[1,0],[2,1/2]], E_2 = [[0,1],[-1,0]]. Representing this basis in the basis B, one gets $E_1 = 1/2\ u_1 + 3/2\ u_2 — 2u_3 + u_4$, and $E_2 = u_2 — 2\ u_3 + u_4$. Thus the standard basis of Span (A) relative to B is: {[[1/2,3/2], [-2,4]], [[0,1],[-2,1]]}.

Question LAI.7 could also be regarded as a computational exercise. One possible strategy is implicitly based on a direct application of the definition of matrix representation of a linear transformation in two bases. Another is derived from the theorem about the relationship between matrix representations of a linear transformation in different bases: if P and Q are transition matrices from standard bases to given bases, then the matrix of the transformation in the given bases is computed as Q^{-1} A P where Q = [[1,1],[0,1]], and P = [[1,1,1],[1,2,1],[1,0,0]], obtaining [[5,-3,0],[1,7,3]]. The student may understand the reasons why these procedures hold, or just remember them; there is no way to discriminate between the two in the solution.

Both Questions LAI.6b and 7 were straightforward computational exercises and there could be no need for conceptual connections worth mentioning. Of course, conceptual understanding of the notion of matrix of a transformation would decrease the risk of "memory overload" and would give the student some means of control over the solution. However, understanding the notion of matrix of a linear transformation itself is conceptually very demanding and achieved only by relatively few students (see Sierpinska, 2000). Students may consider that the theory is not worth the effort necessary to understand it and trust their short term memory in cramming for the final examinations a few days before, ready to forget most of it right after.

Question LAI.8a was yet another typical computational question. Students needed to know *how to* compute the algebraic and geometric multiplicities of the eigenvalues, and *how* to obtain the diagonal form and the basis in which the operator had this diagonal form. They did not need referring to definitions or theorems. When confronted with a problem mentioning "eigenvalues" or "diagonalization" of a matrix or of a linear operator, students normally find the characteristic polynomial, find its roots, solve the corresponding homogeneous equations and construct the bases of their solution spaces to produce eigenvectors in a standard fashion. They can do all that correctly, without the necessity to even understand that an eigenspace is composed of infinitely many eigenvectors. Some students would go through the whole procedure even if, for example, they were only asked to check if a given vector is an eigenvector of a given linear operator. And they would do all the steps even if the theory could afford some convenient shortcuts.

In the case of the present problem, for example, one could proceed as follows. The matrix of T in the standard basis was upper triangular, [[3,1,1],[0,2,0],[0,0,2]]. It was easy to read the eigenvalues off the diagonal without writing down the characteristic polynomial. Likewise, the algebraic multiplicities of the eigenvalues could be read directly from the matrix. A student knowing the relationship between the geometric and algebraic multiplicities did not have to go through the procedure of finding the eigenspaces to state that the geometric multiplicity of 3 must be 1 (since it cannot exceed its algebraic multiplicity, equal to 1, and must be positive). Knowing that geometric multiplicity of 2 was equal to nullity (A — 2I) = 3 — rank (A — 2I), and noticing that rank (A — 2I) = 2 (no calculations necessary), would lead directly to finding that geometric multiplicity of 2 was 2. Already at this point the student could claim that T was diagonalizable, since the geometric multiplicities of its eigenvalues added up to the dimension of the vector space. In the last part of the question, the students were asked to find a basis, in which T had a diagonal matrix. This finally required finding bases of the eigenspaces and solving the homogeneous systems of equations, (A — 3I) X = 0 and (A — 2I) X = 0. Students are normally not penalized for choosing a less economical strategy or doing more computations than necessary. If correct answers are obtained, the student obtains full marks.

Contrary to the previous three questions, Question LAI.8b could not be solved using some standard procedure. The solution required the construction of mathematical objects satisfying certain conditions, having certain properties. Reference to definitions and theorems and connecting them in a reasoning was unavoidable. For example, the student could make a link between the characteristic polynomial and similarity, and think of the fact that the characteristic polynomial, and therefore the set of eigenvalues are invariant under similarity. This could lead to counter-examples such as A = [[1,1],[0,2]] and B = [[2,1],[0,2]]. This was the only question on the test where the ability to disprove a general statement by deriving a contradiction was needed.

Question LAI.8b is a good question to discuss during the course, because the theoretical thinking that it requires is constructed "on the ruins" of the students' quite common belief that similarity of matrices can be verified by row reduction, or that similarity and row equivalence are the same concept. Row reduction and solving equations is what students feel comfortable with; it is their field of practice, and the teacher might capitalize on this practice to engage students with theoretical thinking[9].

Relevance of hypothetical thinking

Hypothetical thinking can be probed in questions that put to the test students' awareness of the assumptions of the theorems they are using, or in questions where the conditions of the existence of a solution have to be discussed. The LAI test did not contain questions that would be specially designed with this goal in mind. On the contrary, in fact. For example, question LAI.3a was so designed that the problem of the existence of a solution did not arise. The matrix A had 3 rows, and this was enough for a solution to exist. Of course, a hypothetically minded student could mention the reasons, why a solution existed in this case. However, this would not raise his or her marks for the problem. If the students were given, say, a 2x3 matrix and asked if there exists a set B for which A would be a matrix of transition from the set A, then hypothetical thinking would have to be used. Question LAI.4b presented a similar (missed) opportunity. If the matrices A and B were not row equivalent, an invertible matrix P such that B = PA would not exist and the problem would not be solvable.

Summary of the analysis of relevance of systemic thinking in solving the LAI examination questions

"Definitional thinking" (in the form of "definitional approach to meanings in algebraic contexts") appeared necessary in 6 (LAI.1b, 3a, 3b, 4b, 6a, 8b), and the activity of proving — in 4 (1b, 4b, 6a, 8b) out of the 14 questions. There were no opportunities for probing the students' tendency to formal categorization, definitional thinking in graphical contexts, axiomatic reasoning, and hypothetical thinking.

Therefore, the examination could be considered as not very demanding with respect to systemic thinking. It was rather surprising that the examination showed so little concern for probing students' understanding of the axiomatic character of the definition of linear transformation — a concept which was, after all, central to the course. The students were only asked (in Question LAI.6a) to show that a given mapping was a linear transformation. The question gave no possibility to discriminate between students, who understood the definition

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[9] See an account of a teaching experiment conducted by J.-L. Dorier, using students' experience with solving systems of linear eqations to introduce the notion of linear dependence/independence (Bartolini-Bussi and Sierpinska, 2000, p. 157).

as postulating a certain general object and those who interpreted the conditions of the definition as "steps in the procedure of solving a certain type of test questions". An example of a question that would allow for such distinction could be the following: Find a linear operator T on R^2 such that T([5, 7]) = [-1,0] and T([10, 14]) = [-2, 0]. Other variants of this question could be used (e.g., where the conditions would lead to a unique linear transformation).

Relevance of analytic thinking

Linguistic sensitivity

The examination was a written one; this fact alone testifies for the importance granted the students' ability to correctly use the formal symbolic notations of linear algebra. We may say that the feature of rigorous interpretation of symbolic expressions was relevant in every question of the test.

As far as the students' sensitivity to mathematical terminology is concerned, it was certainly necessary in understanding the questions themselves: 26 different specialized linear algebraic terms were used in LA I examination questions out of the total of 34 definitions introduced in the course. The students had to recognize if not understand the differences between the meanings of these terms and at least associate them with the correct contexts and typical questions. But this was passive recognition of the terminology. Active use of terminology in full phrases and sentences was required only in 3 questions (LAI.1b, 2b, 8b), where the students were asked to explain or prove their results.

Meta-linguistic sensitivity

Symbolic distance between sign and object

As mentioned in our presentation of the model of theoretical thinking, in the linear algebra courses, students have to get used to the arbitrary, conventional, and temporary character of the meaning of signs. In the LA I test, this symbolic interpretation of signs was most obviously put to the test in three questions (LAI.1b, 3a, 6b). But, in solving some problems (LAI.1a, 4b, 6a, 8a), the student was better off guessing the reference of certain signs from their "shape", as if they were proper names of certain objects. For, in spite of the symbolic character of the signs, lecturers and texts try to reduce the arbitrariness of the symbols and use certain signs always in reference to certain categories of objects. The most popular convention is, of course, the use of the first letters of the Latin alphabet for constants and the last — for variables. It is also quite common, in linear algebra lectures and texts, to denote invertible matrices by the capital P or Q, vectors by lower case u, v, w, vector spaces by upper case U, V, W. Students take this consistency of notation as an implicit assumption and, indeed, they often presume, for example, that the letter P refers to an invertible matrix, even if the assumption is not explicitly made.

In Question LAI.1a, the symbol for the parameter, a "fixed but arbitrary number", is "a", one of the first letters of the Latin alphabet — a traditional symbol for parameters. Putting "x" instead of "a" could have caused some confusion. Students could think that the problem is about functions or transformations, or about an infinite set of vectors, x ranging over all real numbers. Thus, in this question, the student was better off not asking too many questions about the meaning of the letter "a", but automatically interpreting it as "a parameter whose value is to be discussed" in the context of some condition.

Similarly, in Question LAI.4b, interpreting P and Q as names for invertible matrices could facilitate a correct reading of the problem. If the letters C and D were used instead, students might miss the condition of invertibility and try solving an altogether different problem. Likewise, in Question LAI.6a, there was no departure from the habitual "names". "T" is used for transformation and the definition was presented in one of the usual ways: "T(É) = É. The entries of the variable matrix were labeled using the letter "a", used to represent real numbers, and the usual subscripts were used: the first index representing the row and the second — the column. If the question was presented as:

"Let X be the mapping of R_{2x2} into R_{2x2} defined by

$$X : \begin{bmatrix} z_{11} & z_{21} \\ z_{12} & z_{22} \end{bmatrix} \text{a} \begin{bmatrix} 2z_{12} - z_{22} & 2z_{11} + z_{21} \\ z_{12} - 3z_{21} & z_{11} + 3z_{22} \end{bmatrix}$$

Show that X is a linear transformation."

chances are that fewer students would get the solution right. Students might think that X represents a set of mappings and not a concrete mapping, because X is usually a name of a variable. They could also think that the mapping was defined on the transpose of a matrix, and/or that the entries were complex numbers.

Similarly, in Question LAI.8a, seeing letters as names rather than variables was an aid and not an obstacle. Let us imagine the confusion that could be caused just by re-stating the problem as follows:

"Let $T(x_o + x_1 a + x_2 a^2) = (3x_o + x_1 + x_2) + 2x_1 a + 2 x_2 a^2$ ".

On the other hand, in Questions LAI.1b, 3a, 6b the lack of "symbolic interpretation of signs" could mislead the student. In Question LAI.1b the symbols B and C were used to denote elementary matrices instead of the more common symbols E_1 , E_2. Students could fail to notice the assumption and try solving the problem for arbitrary matrices B and C, perhaps even thinking that it is impossible to give a concrete answer to the problem. In Question LAI.3a, nothing more was said about the symbol A except that it was a set of three given vectors. The statement did not say that the symbol represented a basis of R^3. In fact, the set was not a basis. However, transition matrices are usually discussed in the context of change of basis, and some students could think that there is a mistake in the problem or that the problem has no solution. In Question LAI.6b, both lower case and upper case letters were used to denote 2x2 matrices (usually, matrices are denoted using upper case letters). This could be a cause of confusion for students interpreting letters as names.

Sensitivity to the structure and logic of mathematical language

Students' ability to distinguish between a definition and a theorem, a proof and an informal explanation, etc., was not probed in the test. On the other hand, distinguishing between a variable and a parameter ("arbitrary but fixed") in a given problem, identification of implicit quantifiers, discrimination between conditional and biconditional statements, etc. was certainly useful in, at least, a correct interpretation of the questions.

For example, in Question LAI.1a, "a" was assumed to be a "fixed real number", which was intended to mean that a is "arbitrary but fixed" and that, therefore, it could take different values, but, for each value, the set {(a, -1, 5), (3, -2, -a)} contained only two elements and not an infinity of elements, with a ranging over the real numbers. The notion of parameter is not an explicit object of teaching in undergraduate mathematics and students must have some sensitivity to the grammar of mathematical language in order to interpret statements such as in Question LAI.1a. Students lacking this sensitivity may have trouble reconciling the assumption that "a is a *fixed* real number" with the implied suggestion in the second sentence that it may assume different values.

Quantifiers, also not necessarily formally taught to students taking LA I and LA II, were implicit in Questions LAI.1a, 2a, 2b, 5, 6a, 6b, 8a, 8b, but in none of these questions (except perhaps for 8b) was it necessary to be aware of them. For example, formally, Question LAI.1a could read: "True or false: There exists a real number a for which there exist real numbers k and m, not both zero, such that k(a, -1, 5) + m(3, -2, -a) = **0**". But it was possible to solve the problem without being aware of the quantification of variables, treating the variable a as representing an unknown to be determined, and without using any other variables. Dependence of the set could be represented by the condition: the matrix [[a, -1, 5], [3, -2, -a] has rank 1. Considering the cases a =0 and a ›0, the student could come to the conclusion that the set is independent for any value of a.

Understanding the notions of span and basis (Questions LA I.2a, 2b, 5, 8a) involves sensitivity to quantifiers, but the questions could be solved without any subtle considerations of quantifiers. It was not even necessary to understand spans as infinite sets. An intuitive, informal or procedural understanding of the notion of basis was sufficient. Students' awareness of the condition that every vector in a space should be expressible as a linear combination of the basis vectors was not probed. In Question LAI.8b the students had to understand that the statement "matrices of the same rank are similar" means that "any two matrices with the same rank are similar" and not that "some pairs of matrices with the same rank are similar". Thus perhaps LAI.8b was the only question, in which the students' sensitivity to quantifiers was put to the test. Question LAI.4a could be reformulated to also require this kind of logical sensitivity ("Prove or disprove: column equivalent matrices are also row equivalent"), but stated as it was, it did not.

Distinction between conditional and biconditional statements was not explicitly probed although, in their justifications, students could perhaps be "caught" using the false converse of a theorem proved in class or in the textbook. There weren't many occasions for doing that, though. The most obvious occasion was LAI.8b, of course: the student could be answering the question by proving that similar matrices have the same rank. Some occasions were missed, in fact. For example, in Question LAI.2b, where the students had to find whether two spans were equal, the students could use the theorem that if two spaces have different dimensions they cannot be equal. If the two spans had the same dimension, yet were not equal, the question would have the potential to identify students who believed that vector spaces of equal dimensions were *equal*. This is how students sometimes interpret the statement that finite-dimensional vector spaces of equal dimensions are *isomorphic*.

Summary of the analysis of relevance of analytic thinking in solving the LAI examination questions

All questions required sensitivity to mathematical notation and passive recognition of the meaning of terms in the sense of linking them with an appropriate context of use. Active use of terminology was required in 3 questions (LAI.1b, 2b, 8b).

An awareness of the symbolic character of letters used in algebra was probed in three questions (LAI.1b, 3a, 6b). Logical sensitivity was put to the test most obviously in one question (LAI.8b).

Conclusions regarding the relevance of theoretical thinking in the LAI examination

If equal attention was attributed to some n features of theoretical behavior in the examination questions, then the relevance of each feature for high achievement in the examination could be assigned a "weight" of 100/n % of the grade. Let us consider the following six features of theoretical behavior: reflective thinking, definitional approach to meanings, proving, hypothetical thinking, linguistic sensitivity, metalinguistic sensitivity. Then, if equal attention was paid to each feature, the weight of the relevance of each feature would be about 17% of the grade. Based on the preceding analysis of the LA I examination questions, the actual weights attributed to the above mentioned theoretical behavior features departed quite substantially from this kind of even distribution. In the following table (Table 8) we estimated these weights as follows. For example, in solving question LAI.1b four features of theoretical behavior appeared to be relevant: definitional approach to meanings, proving activity, linguistic and meta-linguistic sensitivities. Since the question was worth 6 marks, we assumed that the weight of each of the four features in this question was 6% / 4 = 1.50%. Such calculation was done for each question. The total weight attributed to each of the six TB features in the examination was then estimated by adding up the weights attributed to the feature in each question.

This estimation suggests that the examination focused almost entirely on the students' linguistic sensivity, i.e. their memory and appropriate use of terminology and notation: the weight of this feature was estimated at about 65%, far surpassing the mean value of 17%. All other features were attributed weights below the mean value. The closest to the mean was the weight of the students' definitional approach to meanings, at 13%, followed by the weight of the meta-linguistic sensitivity (i.e. symbolic approach to letters and sensitivity to logic), at 9%.

The weight attributed to proving activity appeared to be less than a half of the mean value, at 8%. Reflective thinking played a very slight role and hypothetical thinking was completely irrelevant in the questions.

		TB-features						
		TB1	TB21	TB22	TB23	TB31	TB322	**SUM in rows**
LAI Question	Marks	Reflective	Definitional	Proving	Hypothetic.	Ling. Sens.	Meta-ling.s.	**(=marks)**
1a	6					6.00		6
1b	6		1.50	1.50		1.50	1.50	6
2a	8					8.00		8
2b	7					7.00		7
3a	6		2.00			2.00	2.00	6
3b	6		3.00			3.00		6
4a	7	3.50				3.50		7
4b	8		2.67	2.67		2.67		8
5	6					6.00		6
6a	7		2.33	2.33		2.33		7
6b	8					4.00	4.00	8
7	7					7.00		7
8a	10					10.00		10
8b	8	1.60	1.60	1.60		1.60	1.60	8
SUM in columns	**100**	**5.10**	**13.10**	**8.10**	**0.00**	**64.60**	**9.10**	**100**

Table 8. Relevance of theoretical behavior in LAI final examination.

A comparison of the above estimation of relevance with the students' theoretical thinking behavior in the interviews will be given in Chapter IV, after a quantitative analysis of this behavior.

ANALYSIS OF THE FINAL EXAMINATION QUESTIONS IN LAII

We reproduce the examination questions below.

The examination questions

Question LAII.1 (5 marks)

Is the following an inner product on R^2? Why?

$(a, b) \cdot (c, d) = ac + 3bd + 3ac + 8bd$

Question LAII.2 (5 marks)

Given that {(1/ˆ2, 1/ˆ2), (1/ˆ2, -1/ˆ2)} is an orthonormal basis for R^2, find an isometry that maps (1/ˆ2, 1/ˆ2) onto (-1, 0).

Question LAII.3 (6 marks)

Let V = R^4 and let W be the span of (2, -2, 1, 1) and (2, -3, 1, -1). Find an orthonormal basis for $W^{\perp}$.

Question LAII.4 (5 marks)

Assume that the trace function (U, V) = trace (U^T V) is an inner product on the space of 2x2 real matrices. Find the norm of $\begin{bmatrix} 1 & b \\ a & 0 \end{bmatrix}$.

Question LAII.5 (7 marks)

(i) Show that the following maps from R^2 to R^2 are projections:

P_1 (a, b) = (a, 3a) P_2 (a, b) = (0, -3a+b)

(ii) Is $P_1 + P_2$ a projection?

Question LAII.6 (5 marks)

Does the matrix $\begin{bmatrix} 0 & 5 \\ 0 & 0 \end{bmatrix}$ have a spectral decomposition?

Question LAII.7 (6 marks)

Find the greatest common divisor of the polynomials a(x) = x + 3, and b(x) = x^2 + 3. Then write this greatest common divisor as a linear combination of a(x) and b(x) with polynomial coefficients.

Question LAII.8 (10 marks)

Let A be the matrix $\begin{bmatrix} -4 & 4 & 16 \\ 3 & 0 & -8 \\ -0.5 & 2 & 5 \end{bmatrix}$.

(i) Is A diagonalizable?

(ii) Can A be written in the form D + N, D diagonalizable, N nilpotent? Why?

Question LAII.9 (9 marks)

What is the Jordan Canonical Form of the following two matrices?

$$A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 3 & 1 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & -1 & 0 \\ 1 & -3 & -1 \\ -7 & 6 & 1 \end{bmatrix}$$

Question LAII.10 (8 marks)

For each $v = (x_1, \ldots, x_n)$ in R^n, a quadratic form q is defined by

$q(v) = X^T A X$ for the given A. Find the rank, index, and the signature of q.

$$A = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$$

Question LAII.11 (5 marks)

Let V be the vector space of all real-valued continuous functions defined on the closed interval [0, 1]. For each $g \in V$, put $h(g) = \int_0^1 g(t)dt$. Determine whether or not h is a linear functional on V. Explain.

Question LAII.12 (15 marks)

Either prove or disprove each of the following assertions about 2x2 matrices over the reals

(a) symmetric matrices are orthogonal

(b) orthogonal matrices are symmetric

(c) orthogonal matrices are diagonalizable

Question LAII.13 (7 marks)

Give an example of two 2x2 real matrices that are similar but not congruent. Explain why.

Question LAII.14 (7 marks)

Give an example of two 2x2 real matrices that are congruent but not similar. Explain why.

Relevance of theoretical thinking in the solution of LAII examination questions

We examine the extent to which theoretical thinking appeared relevant in solving the LAII final examination items. We have organized this section along the theoretical thinking features of our model.

Relevance of reflective thinking

As in the analysis of LAI questions with respect to reflective thinking, we asked if there were any problems in the LAII test that would give the students a chance to temporarily forget about the examination situation and engage in solving an intellectual puzzle. In particular, we looked for open-ended questions, questions with more than one solution, and unpredictable questions (with respect to previous years tests).

There were no open-ended questions in the test, but three questions admitted more than one solution (LAII.2, 13, 14). Three questions were unpredictable with respect to previous years tests (LAII.12, 13, 14). Thus reflective thinking could be considered as relevant in four questions.

Relevance of systemic thinking

We discuss the "definitional" and "proving" features of theoretical thinking simultaneously.

Relevance of definitional thinking and of the activity of proving

We look at each question in turn.

Question LAII.1 required verification, if a given function defined on pairs of vectors from R^2 and values in R was an inner product. It was necessary to refer either to the definition of inner product or to a characterization of inner products in R^n defined by $<u, v> = u^T A v$, where A is a positive definite n x n matrix. But verification by definition could be understood as following three steps of a procedure, and using the characterization — as applying a simple test: represent the product by a matrix, see if it is symmetric, has positive entries on the diagonal and positive determinant. In the given problem the matrix A is a 2x2 diagonal matrix with 4 and 11 on the diagonal, and the solution is trivial. No proving activity was necessary in this case.

At least two approaches to Question LAII.2 were possible. One could take a local geometric perspective and find one obvious isometry satisfying the given condition: say, a rotation about the origin by $3/4\pi$ (or reflection about the line $y = \tan(5/8\pi)x$, i.e. the bisector of the angle between the vectors (ˆ2/2, ˆ2/2) and (-1, 0). This solution could be reached by way of a graphical visualization of the assumptions in a system of coordinates. The understanding of isometry in this case could be based on paradigmatic examples in Euclidean plane geometry: isometries are rotations, reflections, symmetries, translations. This solution is correct, although it was certainly not what the author of the test had in mind. The question did not specify the degree of generality of the solution, did not explicitly ask for all possible solutions and did not require a justification of the answer.

The author of the test had probably in mind an algebraic and general solution: the objects sought were norm preserving linear operators on the vector space R^2 with the ordinary dot product. This interpretation was implied by the fact that an orthogonal basis of R^2 was given in the formulation of the problem (in the geometric approach to the problem, this piece of information was useless). The basis was orthogonal in the sense of the dot product. Students were expected to use one of the following characterizations of isometries in finite dimensional inner product spaces: a linear operator is an isometry iff it preserves the orthogonality of bases; or: a linear operator is an isometry iff its matrix in an orthogonal basis is orthogonal. Using, for example, the latter characterization, one concludes that, in the given orthonormal basis, any isometry F has a matrix with the first column equal to [-ˆ2/2, -ˆ2/2]. If [x, y] denotes the second column, then for the matrix to be orthogonal, it is necessary that -ˆ2/2x -ˆ2/2y = 0 and $x^2 + y^2 = 1$. This implies that x = -y and y = ˆ2/2 or y = -ˆ2/2. Hence there are only two possible isometries satisfying the condition. Since the determinant of the matrix with second column equal to [ˆ2/2, -ˆ2/2] is equal to 1, the isometry is a rotation; the determinant of the other possible matrix being —1, the isometry is a reflection.

The second approach requires definitional thinking and proving activity, but this approach was not necessary for obtaining full marks. The question would have to be formulated differently to force the second approach or to justify not rewarding the geometric local solution by full marks. For example, "find all isometries on the inner product space R^2 with the ordinary dot product that map the vector (ˆ2/2, ˆ2/2) onto the vector (-1,0)".

Question LAII.3 was a standard exercise. A student who knew and was trained in solving this type of problems could do it quite mechanically; there were no surprises in this question. Although there were many possible correct answers (there are many possible orthogonal bases in an inner product space), certain standard procedures eliminated the necessity of reasoning in making choices. The written solution could contain no explanations or justifications, only calculations. For example, as follows (we describe what is being done in the left column, and explain its rationale in the right column):

2 -2 1 1 2 -3 1 -1	The vectors spanning W are put into the rows of a matrix and the matrix is row reduced.
1 0 1/2 5/2 0 1 0 2	The matrix is a matrix of the homogeneous system of equations, which describes the orthogonal complement of W. This description follows not directly from the definition of orthogonal complement but from the theorem that if B is a spanning set of a subspace W then $W^{\perp} = B^{\perp}$.
x = -1/2 z — 5/2 t y = -2t	Parametric representation of the subspace $W^{\perp}$.
z = -2 t = 0 v_1 = (1, 0, -2, 0) z = 0 t = -2 v_2 = (5, 4, 0, -2)	Calculation of a basis of $W^{\perp}$ from its parametric representation by the standard procedure of a convenient substitution of values for the free variables z and t.
$u_1 = v_1$ $u_2 = v_2$ — ((v_2 ¥ u_1)/ (u_1 ¥ u_1)) u_1 = (4, 4, 2, -2)	Application of the Gram-Schmidt orthogonalization process to the basis {v_1 , v_2}
w_1 = (1/ˆ5, 0, -2/ˆ5, 0) w_2 = (2/ˆ10, 2/ˆ10, 1/ˆ10, -1/ˆ10)	Normalization of the orthogonal basis

Question LAII.4 was an even simpler computational exercise. Of course, the student had to know the definitions of the trace of a matrix, and of the inner product — generated norm, to calculate the answer. But the definitions could be treated as formulas, and no proving activity was involved.

Question LAII.5 could be solved as a straightforward application of a definition to verify if a given object (a mapping) belongs to a given category (a projection). In verifying, by definition, if the mappings P_1, P_2, and $P_1 + P_2$ were projections, the student had to understand well the notion of composition of mappings (he or she had to verify if each of the mappings composed with itself was equal to itself). This notion cannot be reduced to a simple procedure and a certain level of conceptual understanding is necessary. A conceptual understanding of the composition of mappings could be avoided in another approach: representing the mappings as matrix transformations (which immediately implies that the mappings are linear) and using the fact that the composition of such mappings is a mapping determined by the product of the respective matrices. In the situation of composition of two distinct mappings, the question of the order of multiplication arises and students who apply this property mechanically often make mistakes. But in this case the matrices had to be squared and the question of order did not arise. In verifying if $P_1 + P_2$ was a projection, the student could represent it by the sum of the matrices, notice that the sum is the 2x2 identity matrix and immediately claim that $P_1 + P_2$ is a projection, identity being a trivial example of projection. In either approach there was some element of definitional thinking and proving activity; neither approach could be reduced to a straightforward application of a procedure.

Question LAII.6 asked whether a given 2x2 matrix had a spectral decomposition. A straightforward approach would be to try to seek the solution directly from the definition (A matrix M has a spectral decomposition iff it can be represented as a linear combination, with coefficients equal to the eigenvalues of the matrix M, of a set of idempotent matrices $E_1, \ldots, E_r$ such that $E_1 + \ldots + E_r = I$, an identity matrix). The given matrix has only one eigenvalue, namely the zero eigenvalue. Thus the spectral decomposition of this matrix, if it existed, would give the zero matrix, which is a contradiction, because the original matrix is not zero. A less straightforward approach would be to use the (very popular) theorem that a matrix has a spectral decomposition if and only if its minimal polynomial is a product of distinct linear factors. It is easy to see that this is not the case for the given matrix (the minimal polynomial is x^2), and therefore it has no spectral decomposition. Another argument could be that the given matrix is nilpotent and non-zero, and hence is not diagonalizable, whereby it cannot have a spectral decomposition. This latter argument could be based on understanding that a nilpotent operator is exactly the part of an operator that is left out when the diagonalizable part is taken away.

In any case, definitional thinking and proving activity seemed to be involved in solving this question.

Question LAII.7 was a purely computational exercise.

Question LAII.8 could be solved in the following way. Using the theorem that a matrix is diagonalizable iff its minimal polynomial decomposes into distinct linear factors, the student could compute the minimal polynomial of A, find that it is equal to $(x + 3)(x - 2)^2$ and conclude that the answer to the first question was "No". The second part of the question was testing if the student remembered the assumptions of the main theorem of the theory about the structure of linear operators: If a linear operator T has a minimal polynomial which decomposes into linear factors (not necessarily distinct) then T = D + N where D is a diagonalizable and N is a nilpotent operator. Since the minimal polynomial of A had the required form then the answer was "yes". However, the question did not specify if the matrix A was regarded as a matrix over **R** or over **C**. Since any polynomial over **C** decomposes into linear factors in **C**, then any linear operator over **C** can be represented as D + N, with D diagonalizable and N nilpotent. A priori, there was no more logical reason to assume that the default interpretation was "A is a real matrix" than that "A is a complex matrix". In the latter case, the student would not even have to refer to the results in part (i) of the question. He or she could just say, "Of course, because every matrix over **C** has such a decomposition". Even if it was found that the minimal polynomial of A is something like $(x^2+3)(x-2)$, the answer to the second part would still be "yes". There was a pragmatic reason, however, to assume that the matrix was a real matrix: the course was restricted to considering real vector spaces and the possibility of developing a more general theory over complex numbers was only mentioned. We conclude that some rather superficial definitional thinking and proving activity was necessary in solving this problem.

Question LAII.9, very typical and well practiced in the homework assignments, would normally be treated by students as a computational problem. They would start by computing the characteristic polynomial of A, obtaining $f(x) = (x - 1)(x - 2)^2$. Since A would be found not to be a zero of the polynomial $(x - 1)(x - 2)$, the students would conclude that the minimal polynomial was the same as the characteristic polynomial. Noticing that the minimal polynomial was not a product of distinct linear factors, they would then conclude that A was not diagonalizable. Eigenvalue 2 has algebraic multiplicity 2, so there must be two 2's on the diagonal of the Jordan canonical form of A. The largest Jordan block corresponding to eigenvalue 2 must be of order 2, since the factor $(x - 2)$ appears with degree 2 in the minimal polynomial. So there was only one possibility: only one 2x2 block with 2's on the diagonal. Since the matrix is 3x3, the remaining block must be 1x1 and this is the block corresponding to 1. A similar "flow chart" would be followed for the matrix B, which would be found to be a nilpotent operator of index 3, whose Jordan form was composed of a single 3x3 block with zeros on the diagonal.

The question did not probe the students' understanding of the Jordan canonical form (JCF) theorem, and what may appear as proving activity was, indeed, only following a procedure, a flow chart, with several "decision boxes". Defnitional understanding would be necessary to make sense of the steps, but there would be no proving activity in this "minimal understanding" approach. Suppose that, for a given matrix, after having found the minimum polynomial, following the procedure would lead to two or more possibilities for the JCF. Some students would then claim that the matrix "has two (or more) JCFs". This would be a clear indication that they had not understood the theorem. But, in the case of the problem in the test, this possibility for probing students' understanding was not given a chance.

Question LAII.10 was related to the notions of rank, index and signature of a quadratic form. The whole point of the concepts of rank, index and signature of a quadratic form is that these numbers are invariant under the change of basis (or "change of variables"). Since these numbers are defined as features of a diagonalized representation of a quadratic form, the practical problem is how to obtain such a representation. With the enormous number of concepts introduced in the course, it is easy to get confused between "diagonalization under similarity" and "diagonalization under congruence" of matrices. One may easily fall into the trap of "overdoing" the problem and start diagonalizing the matrix A by an orthogonal matrix, finding the eigenvectors, using the Gram-Schmidt method to orthogonalize the basis of eigenvectors and even normalizing the basis. Especially, that, in the text used for the course, exactly the same matrix was diagonalized in an example, using, first, an orthogonal matrix P made of eigenvectors. A series of changes of variables were then performed in order to obtain a diagonal representation with only 1'a, -1's and 0's on the diagonal. Such a representation is, of course, very elegant, but it is not necessary if one only wants to know what the index is (and no diagonalization at all is necessary for the computation of the rank: it is enough to row reduce the matrix). There exists an easy algorithm for congruence-diagonalizing a symmetric matrix, but it was not given in the text used for the course. Therefore the students most likely sought to diagonalize the matrix using a basis of eigenvectors. Like in the previous question, students would proceed by following a flow chart of computations and decisions, taking, most probably, a rather uneconomical route, and doing many unnecessary steps. But they would not be rewarded for reasoning, understanding and thus finding shortcuts towards the solution. Thus, definitional thinking would be necessary here, but not the proving activity.

Question LAII.11 was a straightforward application of a definition, but in this case, the exercise was not reduced to following a procedure. Both definitional thinking and proving activity were necessary. But the necessary proving activity was very shallow; it was enough to refer to known facts from calculus. The most important part of the exercise consisted in reducing the problem to these facts from calculus.

Questions LAII.12, 13 and 14 were all quite demanding with respect to definitional thinking and proving activity. Definitions had to be used to construct objects with certain properties; it was not only a matter of verifying if a given object satisfied a given property.

Relevance of hypothetical thinking

We assumed that hypothetical thinking could be given a chance in solving problems where reasoning from default assumptions would lead to contradictions or in discussing the conditions of the existence of a solution. There was one question in LAII where there appeared to be an intention to probe the students' awareness of the assumptions of a theorem, namely, Question LAII.8. This theorem is a fundamental result leading to the Jordan Canonical Form theorem: If the minimal polynomial of a linear operator T on a vector space V over a field F is a product of linear factors over the field F, then T decomposes into a sum of a diagonalizable operatr and a nilpotent operator (Theorem 10.23, p. 344 in Gilbert & Gilbert, 1994). In the question, it was not mentioned what was the field over which the matrix A was being considered. In the course, the default field was the field of real numbers. However, in the context of the eigenvalue theory, the existence of eigenvalues and of the JCF over complex numbers was discussed as well. But, in the test question, the minimal polynomial of the matrix A was $(x - 2)^2(x + 3)$. Thus it could be factored into linear factors over all the subfields of complex numbers, since it was decomposable into linear factors over the minimal subfield, i.e. the subfield of rational numbers. So the answer to part (ii), "Can A be written in the form D + N, D diagonalizable, N nilpotent?" could be an unconditional "yes". There was no need to consider different cases, depending on the field of scalars. The student could just write "yes, because the minimal polynomial factors into linear factors" without mentioning that this holds for any subfield of **C**. It would not be possible to tell, whether the student even thought about the essential role of the field of scalars in the assumptions of the theorem. Had the polynomial been decomposable into linear factors over, say, real numbers but not over rational numbers, or over complex numbers but not over real numbers, then it would be possible to discriminate between students who paid attention to the assumptions of the theorem and those who did not. Thus, the opportunity of a more in-depth probing of students' awareness of assumptions in theorems was missed.

It would not be very difficult to reformulate some of the LAII questions in a way that would direct students towards hypothetical thinking. Let us look, for example, at question LAII.1. The way the rule for the function was written (ac + 3bd + 3ac + 8bd) was somewhat tricky: it suggested that there was something essential about the choice of equal coefficients for the middle terms, while, in fact, there wasn't. One could ask the students if they thought that the equality of the coefficients of the second and third term in the expression was essential for the function to be an inner product[10].

Summary of the analysis of relevance of systemic thinking in the LAII examination questions

Definitional thinking appeared necessary in 11 out of the 14 questions: LAII.1, 4, 5, 6, 8, 9, 10, 11, 12, 13, 14; proving activity — in 7 questions: 5, 6, 8, 11, 12, 13, 14; hypothetical thinking — in one question: 8. In total, some feature of systemic thinking was necessary in 11 questions. The highest relevance was awarded definitional thinking; proving had little priority over following ready-made procedures. Hypothesizing had little significance.

Relevance of analytic thinking

Linguistic sensitivity

As in any written mathematical examination, sensitivity to formal symbolic notations was the sine qua non condition for achieving a high grade. Passive knowledge of terminology also had an important role: the number of new terms that were defined in the course was about 35; 17 of them were used in the formulations of the final examination.

Active knowledge of terminology was probed in questions, in which it was not enough to write a few lines of formal symbolic expressions and finish by a "yes" or a "no". These would be the questions with the instructions to "justify" or "explain" or with the question "why?": LAI.1, 6, 8, 11, 12, 13, and 14.

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[10] We have reasons to believe that the author of the question did not intend it to be "tricky". The intention was to write the expression as "ac + 3bc + 3ad + 8bd". But it was a fortunate mistake, on which one could capitalize to design a good question, going more in depth of the students' understanding.

Meta-linguistic sensitivity

Symbolic distance between sign and object

Symbolic interpretation of signs was necessary in questions LAII.1, 4, 5. In question LAII.1, the formula for the function from R^2 x R^2 to **R** used symbols whose meaning had to be inferred from the assumptions: students had to interpret the letters a, b, c, d as representing arbitrary real numbers (variables) and the symbol "·" as representing "the function" from the formulation of the questions, and not the ordinary multiplication in real numbers. On the right hand side, "ac" etc. had to be interpreted as results of ordinary multiplication in real numbers, and "+" as the ordinary addition in real numbers. On the other hand, the sign "R^2" had to be interpreted as a name of a fixed object: a well-defined set closed under well-defined operations of vector addition and scalar multiplication.

In question LAII.4 a different symbol for inner product was used than in the first question. In question 1, a dot was used; in question 4 — round brackets. In class, the symbol <.,.> was also used for inner products. If Question 4 used the same notation as Question 1, then the definition of the inner product on 2x2 matrices would be written as "U·V = trace(U^T V)". With all these different notations used without warning, the student had to be used to the idea that notation is completely arbitrary modulo certain invariant characteristics. Students had to be sensitive to these invariants. For example, in all notations for the inner product, two variables are used, conveying the idea that the inner product is a binary operation. Another feature of the formulation that could put to the test the students' symbolic distancing was the use of letters "U" and "V" for matrices. This was not very usual in the lectures and in the text; habitually "U" and "V" were used to denote vector spaces.

In Question LAII.5 the choice of letters to denote variables followed the habitus: P_1 and P_2 for projections, a, b for real numbers. However, understanding of letters as variables was necessary in calculating the formulas for the squares of the maps. For example, in calculating P_1 (a, 3a) the letters a and b in the formula of P_1 had to be understood as representing not some specific real numbers but the first and the second coordinates of the variable vector so that if, say, the student was asked to write P_1 (b, a) he or she would not write (a, 3a) but (b, 3b).

In other questions, using the default interpretation based on common usage of a sign was the most economical approach. Especially in question LAII.10 expecting rigorous formalism and explicitness could lead to confusion. The symbol "X" was not defined as having any relation to v. But, being used to the jargon, the default interpretation was: X stands for the coordinate vector of v in some basis of R^n.

Sensitivity to the structure and logic of mathematical language

Quantification of variables was implicit in all questions in one way or another but it was not always necessary to be aware of it. For example, to prove that two mappings are equal (as in question LAII.5, $P_1^2 = P_1$) it is necessary to prove that they have the same domains and ranges and the same values for all elements of the domain. But students often think they have proved the equality by showing that the mappings have the same formulas, and this answer is normally accepted as correct. Sensitivity to quantifiers was more relevant in Question LAII.12. The student had to know that the assertions state that "all matrices of a certain kind are also of some other kind" and not that some matrices of one kind are of this other kind. Also, the student had to have a sense of the nature and role of counterexamples. There was no explicit effort to test the students' distinction between conditional and biconditional statements.

Summary of the analysis of relevance of analytic thinking in the LAII examination

Linguistic sensitivity was, a priori, highly relevant in the LA II test: correct use of mathematical symbolic notation could be probed in each and every question. Active use of mathematical terminology in justifications and explanations was called for in 7 questions (1, 6, 8, 11, 12, 13, 14). Meta-linguistic sensitivity was probed in 4 questions (1, 4, 5, 12).

Conclusions regarding the relevance of theoretical thinking in the LAII examination

Using a similar approach as in our conclusions regarding the relevance of theoretical thinking in the LAI examination, we have attributed "weights" to the features of theoretical behavior that were probed in the LAII questions (Table 9). Like the first examination, also the second one focused on the students' knowledge and use of specialized terminology and notation, but the prominence of this feature in the second examination was much smaller than in the first examination (38 % compared to 65 %). However, part of this loss seemed to be re-gained in the definitional approach to meanings (which also relies on verbal memory): 27% (compared to 13 % in LAI).

LAII Q.	Marks	TB-features TB1 Reflective	TB21 Definitional	TB22 Proving	TB23 Hypothetic.	TB31 Ling.sens.	TB322 Meta-ling.s.	° SUM in rows (=marks)
1	5		1.67			1.67	1.67	5.00
2	5	2.50				2.50		5.00
3	6					3.00	3.00	6.00
4	5		1.67			1.67	1.67	5.00
5	7		2.33	2.33		2.33		7.00
6	5		1.67	1.67		1.67		5.00
7	6					6.00		6.00
8	10		2.50	2.50	2.50	2.50		10.00
9	9		4.50			4.50		9.00
10	8		4.00			4.00		8.00
11	5		1.67	1.67		1.67		5.00
12	15	3.00	3.00	3.00		3.00	3.00	15.00
13	7	1.75	1.75	1.75		1.75		7.00
14	7	1.75	1.75	1.75	°	1.75	°	7.00
SUM in columns	**100**	**9.00**	**26.50**	**14.67**	**2.50**	**38.00**	**9.33**	**100.00**

Table 9. Relevance of theoretical behavior in LAII final examination.

The weight of all other features was below the mean value of 17 % in LAII. Reflective thinking appeared to be slightly more important in LAII than in LAI, its weight being 9 % in LAII compared to 5 % in LAI. Proving activity also had more weight in LAII than in LAI (15 % compared to 8 %). Hypothetical thinking appeared necessary in one question in LAII, while LAI did not seem to probe this feature at all. The meta-linguistic sensitivity had the same level of relative relevance in both examinations.

In Chart 1 we present, in a visual manner, a comparison of the relative weights attributed to the theoretical behavior features in the two examinations, and their relations to the mean value of 17 %.

We end our conclusions with some remarks on the content of the LAII course. In the LAII course, the theory was developed using the language of linear operators: spectral decomposition was an attribute of linear operators mainly, not of matrices. The fact that one could develop a parallel theory for matrices had been mentioned almost as a footnote (p. 345 in Gilbert & Gilbert, 1994). In the lectures, the focus was on the problem of finding suitable simple matrix representations of linear operators and the course culminated in the proof of the Jordan canonical form for linear operators. It was stressed that a linear operator could have different matrix representations in different bases.

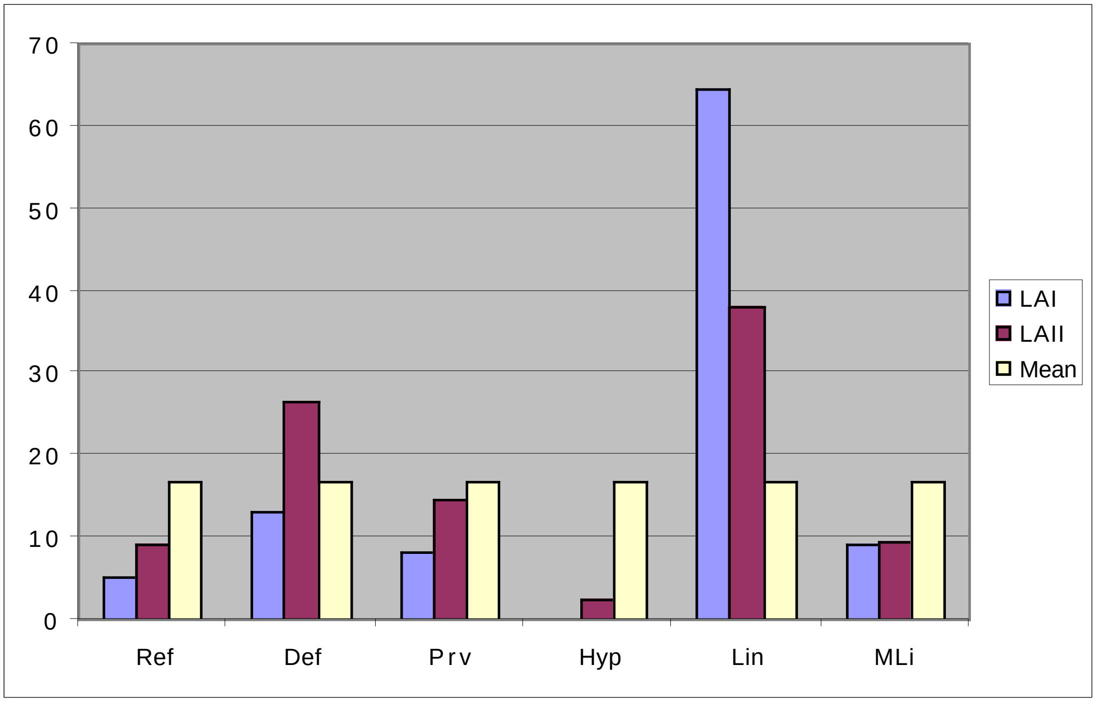

Chart 1. Comparison of the relevance of theoretical behavior features in LAI and LAII examinations.

Thus, in principle, the expression "a matrix [so and so] has a spectral decomposition" used in one of the test questions should have been regarded by the students as an abbreviation of the full statement, "a linear operator represented by a matrix [so and so] in a given basis [so and so] has a spectral decomposition". But, in fact, in their assignments, the students rarely had to construct a basis in which an operator had a diagonal representation and never — to construct a basis in which a given operator would have a non-trivial matrix representation in a Jordan canonical form. And none of the test questions required an awareness of the relativity of a matrix representation of a linear operator in a basis. Students were not even required, in the test, to demonstrate an awareness of the relativity of the representation of a vector with respect to a basis. Eventually, all the students needed to know was some technical knowledge about matrices.

One could perhaps also say that the goal of the course was this technical knowledge about matrices and the theory and language of linear operators was developed only to prove technical theorems about matrices, or rather classes of similar matrices, such as the Jordan Canonical Form theorem (the possibility of finding an especially simple representative of each class). Later in the course, classes of congruent symmetric matrices were studied with the same goal of having each class represented by a diagonal matrix. Once the results were proved, the theory could be rejected, as scaffolding that was no longer necessary. Students were responsible, as usual, for the technically applicable results of the theory but not for the theory itself. If this was indeed the goal of the course, then the use of the "matrix" language was legitimate and should not be considered as full of metonymical abbreviations.

But then one may wonder why this huge theoretical battery was pulled out at all. Surely, the main theorems about matrices could be proved without the structural theory of vector spaces and linear operators, using an analytic-arithmetic approach. Was the structural approach chosen in the name of some greater generality, because some parts of the theory apply also to infinite dimensional spaces and might be useful sometime in the future for some (very few) students?

CHAPTER IV. QUANTITATIVE EVALUATION OF THE STUDENTS' THEORETICAL BEHAVIOR

Let us recall that a student's behavior was coded [1,0] with respect to a given feature of theoretical behavior ("TB feature") if his or her behavior was consistently theoretical, [0,1] — if the behavior was consistently practical, and [1,1] if it was a mixture of the two. In this chapter, these codes have been treated as "scores". This information, in the form of the "Question-by-question" and "Feature-by-feature" tables constituted the basic data for our attempts at a quantitative evaluation of the strengths and weaknesses of the group with respect to theoretical thinking. The "Question-by-question" tables for each question are included in Chapter II and in Appendix I. The "Feature-by-feature" table can be found in Appendix II.

We made many different attempts at ranking the TB features as represented in the group of students. An account of some preliminary approaches can be found in (Nnadozie, 2001; Sierpinska & Nnadozie, 2001). In the following section we present one of our first categorizations.

A FIRST CATEGORIZATION OF THEORETICAL BEHAVIOR FEATURES

Looking at the occurrences of "purely theoretical", "purely practical" and "mixed" behaviors in each of the TB features in the "Feature-by-feature" table, we obtained the following categorization of features:

Category "**τ**":= TB features on which all students scored [1,0]

Category "**π**" := TB features on which all students scored [0,1]

Category "**τ**>>**π**" := TB features on which some students scored [1,0] and some students scored [1, 1] but none scored [0,1]

Category "**π**>>**τ**" := TB features on which some students scored [0,1] and some students scored [1,1], but none scored [1,0]

Category "∅" := TB features on which some students scored [1,0], some students scored [0,1], but none scored [1,1]

Category "∩" := TB features on which some students scored [1,0], some students scored [0,1], and some scored [1,1]

Counting the numbers of features in each category in the "Feature-by-feature" table, the following results were obtained:

Category of TB feature	**Number of features**	**Quantitative and qualitative interpretation**
τ	**1** TB22b (refutation by contradiction)	The overall level of "purely theoretical thinking" in the group was very low. All students spontaneously used some form of contradiction to disprove a general statement.
π	**1** TB22c (axiomatic reasoning)	"Purely practical thinking" was also rare in the group. There was only one feature on which all students behaved as practical thinkers: Axiomatic reasoning. No one in the group engaged in an axiomatic type of reasoning (drawing properties from an axiomatic definition).
$\tau >> \pi$	**5** TB1b (signific.) TB21b (df-alg) TB21c (df-grph) TB321b (rel-grph) TB322a (quant)	Theoretical thinking overshadowed practical thinking in the group on about 1/3 of the features of the model. The group was more inclined to think theoretically than practically about the intrinsic significance of mathematical concepts, and about their meanings both in algebraic and graphical contexts, and were more sensitive to quantifiers than not.
$\pi >> \tau$	**0**	Practical thinking never overshadowed the group's theoretical thinking.
$\varnothing$	**4** TB311a (rigor) TB312a (termin) TB321a (var) TB322b (connc)	Polarization of theoretical vs practical positions occurred on 4 out of the 18 features. There was a polarization of the students' approaches to mathematical language; they were either sensitive to certain subtleties of mathematical language or not, and hints from the interviewers would not provoke much change in their positions.
$\cap$	**7** TB1a (res) TB1c (rel) TB21a (frml-catg) TB22a (proving) TB23a (hypoth) TB322c (frm def) TB322d (impl)	The students were hesitating between theoretical and practical thinking on as many as 7 features: researcher's attitude towards mathematical problems, relational discourse, formal categorization, proving activity, hypothetical thinking, sensitivity to the form of definitions, having a sense of the difference between conditional and biconditional statements. There was no firm adherence to a particular style of thinking in these areas.

Table 10. Categories of TB features

This way of looking at the data led us to view the group as hesitating between theoretical and practical ways of thinking, with more inclination, however, towards theoretical than practical thinking. One very strong aspect of the group's theoretical thinking was their approach to refuting a general statement, by deriving a contradiction. All students spontaneously engaged in reasoning this way. On the other hand, axiomatic reasoning did not belong to the students' spontaneous approaches to proving.

The polarization of students' linguistic sensitivities (rigorous approach to algebraic expressions, careful use of terminology, using letters as variables and not as names of things, sensitivity to logical connectives) was perhaps a result of their varied high school experiences with algebra. By the time the students came to the university, these experiences produced well-entrenched, difficult to change, habits.

This "group portrait" left us with many unanswered questions. For example, we knew that "theoretical thinking overshadowed practical thinking" in the group, but we needed a finer instrument to evaluate the extent of this relation (to be able, for example, to construct a "ranking" of the features on some scale). Moreover, this first analysis did not allow us to make any comparisons with the students' grades in the two linear algebra courses. For this, we needed to evaluate theoretical thinking in the individual students using a scale similar to the grades. This led us to defining certain statistical "indices", on the basis of the two tables of students' scores in the interviews.

Some statistical indices

In the aim of obtaining a more precise measure of the extent to which theoretical thinking overshadowed practical thinking in the group and in the individual students, and of the relation between their thinking and their academic success, we defined a number of "indices". On the basis of these indices we dressed what we called "a theoretical thinking profile" of (a) the group of high achievers in the first linear algebra course (the whole group), and (b) the group of high achievers in both the first and the second course (we label the subgroup by "HiI&II"). A particularly interesting aspect of this profile was the appearance of a "singularity": a student who achieved highly in the two linear algebra courses, but scored in the low range of our indices of theoretical thinking. This singularity led us to question the necessity of theoretical thinking in solving the final examinations, and to analyzing the final examinations questions from this point of view. This analysis was done in the previous chapter.

Group behavior indices

We decided to compute three "indices" of the group of 14 students' behavior with respect to the 18 features of theoretical behavior. These indices were labeled: "Expectation" (E), "Group Theoretical Thinking Tendency" (gtt), and "Capability" (C). We define these indices below.

For a given feature, let t, p, h denote the number of students who scored [1,0], [0,1], and [1,1], respectively. In other words, t, p and h could be defined as the numbers of students for whom the probability of behaving in a theoretical way was 1, 0 or 1/2, respectively. Let n be the total number of students. Then $n = t + p + h$. In our case, n was equal to 14.

The expectation E of theoretical behavior on a given feature by a randomly chosen student was then defined as follows:

$$E = (1\cdot t + 0\cdot p + .5\cdot h)/n = (2\cdot t + h) / (2\cdot n)$$

This was not the first index we thought about. This measure downplayed the role of practical thinking in mathematical thinking, which we assumed to be significant. We were more interested in the "ratio" between theoretical thinking and mathematical thinking in general whether theoretical or practical. We wanted this ratio to express the group's disposition to think theoretically and not just the probability that a student would think theoretically, as if no other ways of thinking counted in mathematics. We defined the "theoretical thinking tendency" of the group with respect to a given feature as the ratio of the number of occurrences of theoretical thinking (t + h, or the number of those who scored [1,0] or [1,1]) to the number of occurrences of theoretical or practical thinking (t + h + p + h, or the number of those who scored [1,0], [0,1] or [1,1]) in the group. We labeled it "gtt" ("group theoretical tendency"):

$$gtt = (t+h)/(t+2\cdot h+p) = (t+h)/(n+h)$$

The indices E and gtt do not represent the group's potential or capability to behave theoretically. Capability for theoretical thinking can be revealed with specially designed, conflict provoking problem situations, or hints from the teacher (interviewer) in a student who spontaneously chooses a practical approach. Thus this capability could be measured by the ratio of the number of students who scored [1,0] (t) (spontaneous theoretical behavior) or [1,1] (h) (prompted theoretical behavior) to the total number n of students (14 in our case). We denoted this index by "C":

$$C = (t + h)/n$$

In the language of the variables t, p and h, the categories of features discussed in the previous section could be described as follows:

τ	$t > 0$ $p = 0$ $h = 0$
π	$t = 0$ $p > 0$ $h = 0$
$\tau >> \pi$	$t > 0$ $p = 0$ $h > 0$
$\pi >> \tau$	$t = 0$ $p > 0$ $h > 0$
$\varnothing$	$t > 0$ $p > 0$ $h = 0$
$\cap$	$t > 0$ $p > 0$ $h > 0$

Relations of magnitude among E, gtt and C, and their interpretation in the case of our interviews

h = 0

If h, the number of students who behaved both theoretically and practically with respect to a TB feature, is equal to zero, then all three indices are equal to the proportion of purely theoretically thinking students in the group:

$$h = 0 \;\Rightarrow\; E = gtt = C = t/n$$

The situation where h = 0 was the case where students behaved either consistently theoretically or consistently practically, but did not change their way of thinking during the interview. This could mean that a particular behavior was well consolidated, and could be difficult to change. Some behaviors could thus function as obstacles.

In the case of the group of students that we interviewed, such potential source of obstacles was the relatively important phenomenon of sloppiness with respect to algebraic expressions and mathematical terminology (TB311a (rig), t = 6, p = 8, h = 0; TB312a (term), t = 6, p = 8, h = 0; in both cases, t < 7, p ‡ 7, h = 0).

h > 0

If the number h of students who either hesitated between two ways of thinking or changed their way of thinking during the interview was positive (h > 0), then the index C was always more "optimistic" or greater than both E and gtt:

$$E = (2t + h)/(2n) < (2t + 2h)/(2n) = C$$

$$gtt = (t + h)/(n + h) < (t + h)/n = C$$

The index C points to the cognitive potential of the group and to the possibilities of successful didactic intervention.

Particularly significant, from the didactic point of view, could be the cases where the indices E and gtt were at medium or low levels but $C = 1$. They could point to areas with a great potential of improvement. In the case of our interview, these areas were: analytic understanding of graphical representations of functions and sensitivity to quantifiers (TB21c, 321b, 322a). We could say that these domains of analytical thinking were within the group of students' "zone of proximal development" (ZPD) (Vygotsky, 1987).

Another way of looking for areas with a potential for improvement of theoretical thinking could be to look at cases with a large h and small t and p. If h grows close to n, then both t and p become small, and E and gtt tend to 1/2, while C tends to 1. This is a situation, where the students were as likely as not to behave in a theoretical way, but were highly capable of behaving theoretically, with some hint or help. In the interviews, these cases covered proving activity (TB22a, $t = 1$, $p = 3$, $h = 10$), hypothetical thinking (TB23a, $p = 2$, $p = 1$, $h = 11$), and logical sensitivity (TB322a, $t = 4$, $p = 0$, $h = 10$; TB322d, $t = 4$, $p = 1$, $h = 9$). Again, this could identify areas within the group of students' ZPD.

Relations between E and gtt

E < gtt. Let us now look at the relations between E and gtt. If $h = 0$, then $E = gtt$. Assume now h ‡ 1. Then $E < gtt$ if and only if $n > h + 2t$, which is equivalent to $p > t$. Thus $E < gtt$ if and only if $p > t$. Thus if the number of practically thinking students on a feature is greater than the number of theoretically thinking students, with the number of "flexible" students being non-zero, then the Expectation index is more pessimistic than the group theoretical thinking tendency: the overall group's tendency to behave theoretically is greater than the likelihood that a randomly chosen student from the group will behave theoretically on a feature. There were two instances of $p > t$ and $h > 0$ in our interview, and these were the researcher's attitude (TB1a, $E = 0.46$, $gtt = 0.48$, $C = 0.71$) and proving activity (TB22a, $E = 43$, $gtt = 0.46$, $C = 0.79$). However, the differences between E and gtt in these cases were slight.

E > gtt. This relation occurs with $t > p$ and $h > 0$. The likelihood of a randomly chosen student to behave theoretically was greater than the group's tendency in the case of 10 TB features: TB1b (appreciation of the intrinsic significance of mathematical concepts); TB1c (relational discourse); TB21a (formal categorization); TB21b,c (definitional approach to meanings); TB23a (hypothetical thinking); TB321b (interpretation of graphs as representations of relationships between variables rather than as shapes), TBB322a,b,c (sensitivity to quantifiers, form of definitions and distinction between a conditional and a biconditional statement). These were the cases where $t > p$ and $h > 0$, i.e. the number of students behaving purely theoretically exceeding the number of students behaving purely practically, with the number of those able to switch from practical thinking to theoretical thinking greater than 0. In 5 of these features, $C = 1$, implying the students' high ability to benefit from the interviewer's hints or from reflection triggered by a challenging problem. This observation corroborates and refines our earlier conclusion that, generally in the group, theoretical thinking overshadowed practical thinking.

E = gtt. This would occur with $p = t$, and then $E = gtt = 1/2$. In the interviews, this case did not occur.

Individual behavior indices

We used analogous indices to characterize individual students' ways of thinking. As we wanted to compare the values of these indices with the students' grades in the linear algebra courses, we represented them as percentages (grades being represented as percentages as well).

We calculated the individual behavior indices for the Feature-by-Feature table (F-b-f) and for the Question-by-Question table (Q-b-q). For Q-b-q, the indices were calculated for each question and then their weighted averages were calculated to represent a student's thinking in the whole interview. The weight attributed to an index on a question was taken to be the ratio of the number of TB features revealed in this question to the total number of TB features revealed in all questions. In our case this total number of TB features was 25. The number of distinct TB features was 18, but some features appeared in more than one question.

We first define the variables it, ip, and ih (for F-b-f) and it_q, ip_q, and ih_q (for Q-b-q).

it := number of features in the interview as a whole (F-b-f table) on which the student's behavior was coded as [1,0].

ip:= number of features in the interview as a whole (F-b-f table) on which the student's behavior was coded as [0,1].

ih := number of features in the interview as a whole (F-b-f table) on which the student's behavior was coded as [1,1].

it_q := number of features in a given question q of the interview (Q-b-q table) on which the student's behavior was coded as [1,0].

ip_q := number of features in a given question q of the interview (Q-b-q table) on which the student's behavior was coded as [0,1].

ih_q := number of features in a given question q of the interview (Q-b-q table) on which the student's behavior was coded as [1,1].

Let f denote the number of features revealed in the whole interview (f = 18 in our case). Then it + ip + ih = f.

If we denote by f_q the number of features revealed in a question then, in a particular question, $it_q + ip_q + ih_q = f_q$.

We now define the indices.

iE% := (2it + ih)/(2f) * 100, measuring the Expectation that the individual student will behave theoretically on a randomly chosen TB feature from the given set of f features revealed in an interview.

itt% := (it + ih)/(f + ih) * 100, measuring the individual theoretical thinking tendency of a student in an interview.

iC% := (it + ih)/f * 100, measuring the individual student's capability to think theoretically in an interview, given favourable circumstances.

iE_q % := $(2it_q + ih_q)/(2f_q)$ * 100, measuring the Expectation that the individual student will behave theoretically on a randomly chosen TB feature from the given set of f_q features revealed in a given question q in an interview.

itt_q % := $(it_q + ih_q)/(f_q + ih_q)$ * 100, measuring the individual theoretical thinking tendency of a student in a given question in an interview.

iC_q % := $(it_q + ih_q)/f_q$ * 100, measuring the individual student's capability to think theoretically in a given question in an interview, given favourable circumstances.

Let now r be the number of questions in an interview, and $f_1, \ldots, f_r$ denote the numbers of TB features revealed in questions $q_1, \ldots, q_r$, respectively. Let $f_1 + \ldots + f_r = s$. Let iE_k %, itt_k %, iC_k % denote the indices characterizing an individual student in question k. Then the "weighted average" indices obtained from the Q-b-q table are defined as follows:

wiE% := $f_1/s \cdot iE_1\% + \dots + f_r/s \cdot iE_r\%$

witt% := $f_1/s \cdot itt_1\% + \dots + f_r/s \cdot itt_r\%$

wiC% := $f_1/s \cdot iC_1\% + \dots + f_r/s \cdot iC_r\%$

In our case, r = 7, s = 25, whence,

$wiE\% = 3/25 \cdot iE_1\% + 2/25 \cdot iE_2\% + 6/25 \cdot iE_3\% + 3/25 \cdot iE_4\% + 4/25 \cdot iE_5\% + 4/25 \cdot iE_6\% + 3/25 \cdot iE_7\%$

$witt\% = 3/25 \cdot itt_1\% + 2/25 \cdot itt_2\% + 6/25 \cdot itt_3\% + 3/25 \cdot itt_4\% + 4/25 \cdot itt_5\% + 4/25 \cdot itt_6\% + 3/25 \cdot itt_7\%$

$wiC\% = 3/25 \cdot iC_1\% + 2/25 \cdot iC_2\% + 6/25 \cdot iC_3\% + 3/25 \cdot iC_4\% + 4/25 \cdot iC_5\% + 4/25 \cdot iC_6\% + 3/25 \cdot iC_7\%$

RELATIONS BETWEEN THE STUDENTS' THEORETICAL THINKING AND THEIR ACADEMIC SUCCESS

Students as individuals

As we compared the individual weighted average indices of theoretical thinking with the students' grades in the two linear algebra courses, we found that high grades in the first linear algebra course did not correlate with the students' theoretical thinking (Table 11).

Question-by-question table

WEIGHTED AVERAGES °				GRADES °	
wiE%	witt%	wiC%	STUDENTS	LA I	LA II
38.00	38.93	48.00	O1	92	70
36.00	35.20	48.00	O2	82	50
90.00	**88.00**	**96.00**	**O3**	**83**	**90**
46.00	47.60	60.00	O4	82	63
66.00	63.60	80.00	V1	82	70
78.00	**74.80**	**88.00**	**V2**	**98**	**85**
74.00	**71.20**	**84.00**	**V3**	**82**	**80**
54.00	54.03	64.00	V4	93	60
74.00	73.40	80.00	S1	85	°
52.00	51.40	64.00	S2	88	°
62.00	**60.80**	**72.00**	**S3**	**92**	**85**
72.00	71.07	80.00	S4	88	77
50.00	**50.40**	**60.00**	**N1**	**100**	**90**
72.00	**68.60**	**84.00**	**N2**	**100**	**85**
			cor Gr/wiE%	-0.01	0.69
			cor Gr/witt%	-0.02	**0.70**
			cor Gr/wiC%	-0.02	0.66

Table 11. Weighted indices obtained from the Question-by-question table and correlation with the students' grades in LAI and LAII.

The correlation was positive with the grades in the second course, the correlation being the highest for the weighted individual theoretical thinking tendency (witt%), obtained from the Question-by-question table. But even in this case it was not very high (.70). This suggested that perhaps what we understood by theoretical thinking was not indispensable to obtain high grades in the linear algebra courses. This hypothesis was supported by our analysis of the final examination questions in the previous chapter. The correlation between the values of individual indices obtained from the Feature-by-feature tables and LAII grades was 0.68 for E% and gtt% and 0.57 for C% (Appendix II). The correlation was null with LAI grades. Thus the individual indices obtained from the F-b-f tables had an even lower value as predictors of the grades than the weighted average indices obtained from the Q-b-q tables.

In Table 11 we highlighted the results of the students who achieved highly (‡ 80%) in both courses (the HiI&II subgroup: O3, V2, V3, S3, N1, N2). It was interesting to look at the correlation between the grades of the subgroup HiI&II and the weighted averages of their indices in the Question-by-question table (Table 12).

HiI&II individual indices

WEIGHTED AVERAGES				GRADES	
wiE%	witt%	wiC%		LA I	LA II
90.00	88.00	96.00	O3	83	90
78.00	74.80	88.00	V2	98	85
74.00	71.20	84.00	V3	82	80
62.00	60.80	72.00	S3	92	85
50.00	**50.40**	**60.00**	**N1**	**100**	**90**
72.00	68.60	84.00	N2	100	85
			cor Gr/wiE%	-0.14	0.54
			cor Gr/witt%	-0.17	0.53
			cor Gr/iC%	-0.15	0.49

Table 12. Correlation between the grades of high achievers in both courses and their weighted indices. The discrepancy between the indices of the student N1 and his grades is highlighted.

If we look for those features of theoretical behavior on which all students who achieved high grades in both courses scored [1,0], then there were only three of them: definitional approach to meanings in algebraic contexts, refutation of a general statement using some form of contradiction and sensitivity to logical connectives. This might suggest that these features were sine-qua-non conditions for high achievement in both courses. Our analysis of the examination questions confirms the relative importance of the definitional approach to meanings, especially in the second course. Proving was less important in general, but refutation of a general statement was necessary in one question in LAI (8b) and in the three parts of question LAII.12, and the construction of a counterexample was explicitly requested in LAI.4a and LAII.13, and LAII.14. Metalinguistic sensitivity was not given the highest weight in the examinations but it was not completely unimportant, either.

We represent the grades in LA II and the witt% indices of the HiI&II subgroup in Chart 1 below. Table 12 and Chart 1 highlight an important singularity in the HiI&II subgroup: the student N1 whose grade in LA II was 90%, yet whose theoretical thinking tendency appeared quite low (wiE%, witt% ~ 50%, wiC% = 60%).

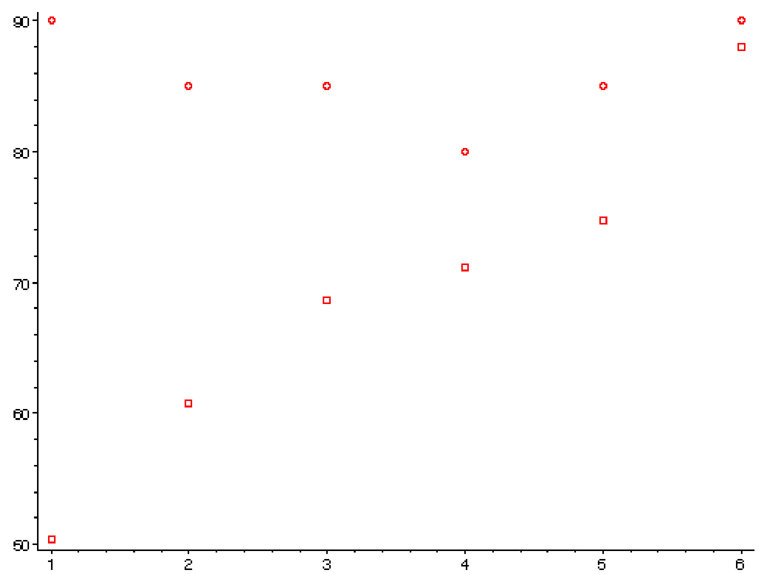

Chart 2. Plot corresponding to grades in LA II (circles) and witt% indices (boxes) in the HiI&II subgroup, in the following order: N1, S3, N2, V3, V2, O3.

What were the features on which student N1 scored [0,1]? Axiomatic reasoning, researcher's attitude towards mathematical problems, rigorous approach to algebraic expressions, sensitivity to mathematical terminology, sensitivity to the form of definitions, and distinction between conditional and biconditional statements. It appears that he did not need these features of behavior to achieve his 90% on LA II. This contrasts with our estimation of the relevance of linguistic sensitivity in the final examinations at 65% in LAI and 38% in LAII.

There was yet another student in the group whose witt% index was close to 50%: S2. She was one of the two students who did not take the second linear algebra course. She did not take the course because she changed her program of studies and went to study psychology. She scored [1,0] on all the features on which N1 scored [1,0] (relational discourse, definitional approach to meanings, disproving a general statement by a contradiction, interpretation of letters as variables, interpretation of graphs as representations of relationships, sensitivity to logical connectives). She was more theoretically inclined than N1 in formal categorization and in distinguishing between conditional and biconditional statements.

The most striking difference in the observable demeanor of the two students was that N1 appeared very sure of himself and S2 did not. A less explicit difference was that it seemed that N1 could make do with just enough understanding to effectively solve a typical test problem, while S2 had a desperate need to understand every detail of the theory. She explained to us that she needed explicitness and rigor in using mathematical notations and language, while her teachers seemed to take it for granted that the students would understand from the context in what meaning a given variable was used. She had an ability to reflect on her own understanding, which certainly relieved her of some of her misconceptions, but the time this required was not helping her in keeping up with the tight schedule of the actuarial mathematics program. During the interview, she shared with us her experience with understanding the concept of linear independence. She was able to diagnose her difficulty as having its source in her uni-dimensional concept of variable. She was used to one-dimensional variables in her first calculus classes, which she took in her commerce program at the "cegep"[11]. Now at the university, in the linear algebra course, vectors were brought in as variables and they were multi-dimensional. She told us that her first understanding of the variables v_1, É, v_n in the equation $a_1 v_1 +$ É $+ a_n v_n = 0$, was that they represented the components (or entries) of one single vector, not a set of vectors.

> S2: I can see the thing [*the difficulty of the student mentioned in question 2 'linear independence definition'*]. I don't know if this is of any help to your study, but I have to say that that concept for me was very hard toÉto catch on, I didnÕt quite understand what they were saying, at first, with the vectors, because, to me, v was always oneÉlike, a variable was always one [*dimensional*], so it was hard for me

[11] "Cegep" refers to "Coll ge d'Enseignement G n ral et Professionel", an educational institution between high school and university in the province of Qu bec in Canada.

to catch on to the fact that it was two. So I had a lot of trouble to finally get into what that sentence, that mathematical sentence meant.

AS: Mhm.

S2: So I can understand whyÉ

AO: Can you explain a little bit what you mean by one and two?

AS: One variable, you said?

S2: Oh, okay, 'cause, I don't know, I wasn't from a science domain, so I wasnÕt really familiar with linear equations and suchÉIt wasn't really an important aspect of the commerce side of everything, so when IÉMy original concept of a variable was that it has one value, and not, like, a vector, which can have[É] So it was hard, because when IÉI kind ofÉThe way I saw it in my head was that v_1 was, like, all the vÕs were one huge vector, and that was the first element, and it took me a long time to stop thinking that wayÉ[É] I mean, once I got it, it was obvious, but I had a hard timeÉI don't know, maybe it's different because I came from a commerce stream into the math, and it's people with sciences, it's a bit different. [É]You see, I was, basicallyÉCalculus, for me, easy (*clicks her fingers*), like I had no problem at all, I had no problem with Calculus, I was always very good at Calculus, and I kind of assumed that all math was like Calculus, and thenÉLinear Algebra in terms ofÉin terms of commerce, at cegep level, itÕs basically, multivariable Calculus I. So when I came into this Linear at University, I found it was very much different, so I tried to keep myself afloat by reading the book, and I found the book extremely confusing, because, they are using all these letters but they are using them interchangeably for other things, or at least that's how I felt. Because they use, likeÉMy teacher would useÉu and v, and all the u's wereÉumÉOkay, in one time, he used the u's and the v's to like, say, this one was v, and then this one was u. And then the next day, heÕd use them in the oppositeÉlike, u and v for the opposite, like, all of a sudden, they were their own categories. So, you know, it was hard, because u and v meant two different things, and I guess, I am the type of a person that gets stuck onÉif this is u, this is u, and that's all that u can be. And the fact that they were changing what u meant, to me, wasÉ

Unlike N1, S2 was aware of and bothered by the fact that her natural logic did not always match the mathematical logic. She was aware of the symbolic character of letters in algebra, their conventionally assigned meaning, but it was not coming "naturally" to her; she had to make herself aware of that each time she was reading a formula. She was also very uncomfortable with the lack of control she often experienced over her reasoning and its outcome in writing proofs. She often used the word "secure" and it seemed that she craved for the feeling of security, and she could not find it in mathematics. For example, in the question on the "linear dependence typo", she suggested "fixing" the problem by adding an assumption that the initial vectors are independent. The reason she gave was that it would make her feel more secure:

> I think that would just make me feel more secure about it, because IÉIÉthe way I was trained was, like, the independenceÉokay, if you know those were independent then it’s not going to mess up anything else, so youÕve got that and then you know how to set up your scalar equation. (S2)

This observation could lead us to conclude that one of the most important necessary conditions of success in mathematics is self-confidence (more than theoretical thinking). More precisely: one should not be afraid of not understanding and getting it wrong but learn to survive on the little understanding one has and hope that, with time and experience, things will get clearer.

Students as a group

We now look at the students as a group. We look at the whole group and the HiI&II subgroup and dress the "group portraits" of their theoretical thinking strengths and weaknesses. The data basis for this analysis are the F-b-f tables. We rank the features of theoretical thinking as revealed in the interviews along the E, gtt and C indices. We say that an index is at a "high level" if it is ‡ 0.80, at a "medium level" if it is between 0.60 and 0.79, and at a "low level" if it is below 0.60.

Ranking of the main categories of features

We first rank the main categories of TB features: Reflective, Systemic and Analytic. To do this, we compute the averages of the indices obtained for the particular aspects of each of these main categories. The results are in Table 13.

For the whole group, one main feature - Systemic thinking — fell below 0.60 on one index, namely the gtt (group’s tendency to think theoretically). For HiI&II, all indices were above 0.60. In both the whole and the HiI&II groups the capability index was the highest but while it was the strongest on Reflective thinking in the whole group, in the HiI&II subgroup, capability was the strongest in the area of Systemic thinking, followed by Analytic thinking and Reflective thinking coming the last. It may well be that Reflective thinking, with its researcher’s attitude especially, is not very helpful when it comes to performance on limited time tests and solving standard exercises.

Both in the whole group and the HiI&II subgroup only the capability index came within the high level range. The group tendency (gtt) and Expectation (E) remained at the medium level (or low in one case: Systemic thinking, gtt = 0.58 in the whole group). With respect to the group tendency index, Reflective thinking came up the strongest for the whole group (.64), while for the HiI&II subgroup, Analytic thinking turned out to be the strongest (0.68). Expectation ordered the features in the same way for both groups: Reflective, Analytic and Systemic thinking.

Average values of indices on main categories of features

		E		gtt		C	
°	TB	Whole gr.	HiI&II	Whl.gr.	HiI&II	Whl.gr.	HiI&II
1	1a (res)	0.46	0.67	0.48	0.67	0.71	0.83
2	1b (sig)	0.82	0.83	0.74	0.63	1	0.75
3	1c (rel)	0.71	0.75	0.67	0.63	0.86	0.75
Av. REFLECTIVE		**0.67**	**0.75**	**0.63**	**0.64**	**0.86**	**0.78**
4	21a (cat)	0.68	0.75	0.65	0.71	0.79	0.83
5	21b (def-a)	0.86	1	0.78	1	1	1
6	21c (def-g)	0.75	0.83	0.67	0.75	1	1
7	22a (prv)	0.43	0.58	0.46	0.55	0.79	1
8	22b (ref)	1	1	1	1	1	1
9	22c (ax-r)	0	0	0	0	0	0
10	23a (hyp)	0.54	0.58	0.52	0.55	0.93	1
Av. SYSTEMIC		**0.61**	**0.68**	**0.58**	**0.65**	**0.79**	**0.83**
11	311a (rig)	0.43	0.5	0.43	0.5	0.43	0.5
12	312a (term)	0.43	0.33	0.43	0.33	0.43	0.33
13	321a (var)	0.71	0.83	0.71	0.83	0.71	0.83
14	321b (grph)	0.75	0.83	0.67	0.75	1	1
15	322a (quan)	0.64	0.75	0.58	0.67	1	1
16	322b (con)	0.71	1	0.71	1	0.72	1
17	322c (f-def)	0.68	0.83	0.62	0.83	0.71	0.83
18	322d (imp)	0.61	0.58	0.57	0.56	0.93	0.83
Av. ANALYTIC		**0.62**	**0.71**	**0.60**	**0.68**	**0.74**	**0.79**

Table 13. Group indices on the TB features obtained from the Feature-by-feature table for the whole group and the HiI&II subgroup

Ranking of all features

We now look at the rankings of the TB features according to the E, gtt and C indices in the whole group and the HiI&II subgroup (Appendices II and III). The first difference that strikes the eye is the much greater number of features in the high level range for the subgroup HiI&II in each ranking. For example, the probability that a randomly chosen student from the whole group would behave theoretically (E) was high on three features, while it was high on as many as 8 features for the HiI&II subgroup. The group tendency was a harsher measure and the whole group's tendency was high on 1 feature only; it was high on 5 features in the HiI&II subgroup. The respective numbers for the groups' capability (the most tolerant measure) were

9 and 13. This last outcome suggests an important potential for theoretical thinking in the subgroup of high achievers in both courses. This potential might have remained largely unused, in view of the low exigency with respect to theoretical thinking in the examination questions.

In the whole group the gtt index was leading in the number of low scoring features with 8, followed by 6 low scoring features on the E index and 3 low scoring features on the capability index C. The situation was similar in HiI&II, but there were as many low scoring features for gtt as for E (6) and again 3 for C.

In both groups, gtt was less than or equal E on all features, implying that p † t or that practical thinking did not prevail over theoretical thinking in the group.

Using now gtt as the least generous measure, we identify the strongest and the weakest points in the HiI&II subgroup (Table 14).

	HiI&II - Ranking according to gtt		
TB	E	gtt	C
21b (def-a)	1	1	1
22b (ref)	1	1	1
322b (con)	1	1	1
321a (var)	0.83	0.83	0.83
322c (f-def)	0.83	0.83	0.83
21c (def-g)	0.83	0.75	1
321b (grph)	0.83	0.75	1
21a (cat)	0.75	0.71	0.83
322a (quan)	0.75	0.67	1
1a (res)	0.67	0.67	0.83
1b (sig)	0.83	0.63	0.75
1c (rel)	0.75	0.63	0.75
322d (imp)	0.58	0.56	0.83
22a (prv)	0.58	0.55	1
23a (hyp)	0.58	0.55	1
311a (rig)	0.5	0.5	0.5
312a (term)	0.33	0.33	0.33
22c (ax-r)	0	0	0

Table 14. Ranking of the TB features in the HiI&II subgroup, according to the gtt index.

High achievers in both courses appeared to be strong on the definitional approach to meanings in algebraic contexts, disproving a general statement by deriving a contradiction, sensitivity to logical connectives, interpreting letters as variables and sensitivity to the form of definitions. The whole group was strong only on disproving a general statement by deriving a contradiction. This might imply that these features were important for high achievement in the linear algebra courses, with the feature of refutation by contradiction not being sufficient for high achievement.

High achievers in both courses scored low gtt on distinguishing between a conditional and a biconditional statement, proving activity, hypothetical thinking, rigorous approach to mathematical expressions, sensitivity to mathematical terminology and axiomatic reasoning. The whole group scored low on two more features, namely sensitivity to quantifiers and researcher's attitude. The HiI&II subgroup scored in the middle range on these latter two features. This could imply that these features may be an asset for high achievement, even if they are not a necessity.

Students' theoretical thinking profile and the final examinations

In Chapter III we have analyzed the LAI and LAII examination questions by trying to estimate the importance that these questions awarded the various theoretical behavior features identified in the interview. We expressed these values in percentages of the grade. Now, we express them as numbers in the interval from 0 to 1, and compare them (Table 15) with averages of the E, gtt and C indices obtained in the categories of Reflective (TB1a,b,c), Definitional (TB 21a,b,c), Proving (TB22a,b,c), Hypothetical (TB23a), Linguistic sensitivity (TB311a, 312a), Meta-linguistic sensitivity (321a,b, 322a-d) for the whole group (the values from which the averages have been obtained can be found in Table 14).

	Averages of indices				
TB	E	gtt	C	LA I	LAII
Reflective	0.67	0.63	0.86	0.05	0.09
Definitional	0.76	0.7	0.93	0.13	0.27
Proving	0.48	0.49	0.6	0.08	0.15
Hypothetical	0.54	0.52	0.93	0	0.03
Ling.sens.	0.43	0.43	0.43	0.65	0.38
Meta-lin.sens	0.68	0.65	0.84	0.09	0.09
			cor LA /E	-0.52	-0.24
			cor LA /gtt	-0.55	-0.27
			cor LA /C	-0.81	-0.65

Table 15. Indices of the interviewed students' theoretical thinking tendencies and the a priori values attached to the importance of theoretical behavior features on the LA I and LA II final tests.

The correlation between the relative relevance of the TB features in the test and the high achievers' theoretical thinking tendencies measured in the three ways is negative and relatively highly negative with respect to the capability index.

If we consider the feature which appeared to be highly valued in the examinations (linguistic sensitivity), and recall that the student N1 scored low on the rigorous approach to algebraic expressions and on sensitivity to mathematical terminology, then we may conclude that we may have overestimated this feature in our a priori analysis of the test. Perhaps the instructors, marking the students' work, were very tolerant with respect to the way the students presented their solutions and explanations. For many Concordia University students, English is not their first language (many have done all their pre-university education in French or in some other language) and it is common to be very tolerant when marking the students' written explanations in mathematics. Students are also not overly penalized for lack of notational rigor. It is understood that the students are nervous during the exam, waste much time making mistakes in calculations, and don't always have the time to re-write their solutions in a more rigorous way.

The next feature in order of importance in the examinations appeared to be the definitional approach to meanings in algebraic contexts. Let us recall that for the HiI&II subgroup, all three indices were equal to 1 on definitional approach to meanings in algebra (there were no categorization nor graphical items in the tests). These students also scored highly on interpretation of letters as variables. They could all refute a general statement by deriving a contradiction. The indices for proving a general statement and axiomatic reasoning, hypothetical thinking, and researcher's attitude were medium to low and very low for this subgroup, but these features did not appear to matter so much for passing the tests, either.

In sum, it did not seem to be worthwhile to behave as a researcher rather than as a student in the LAI test. It was a waste of time to engage in hypothetical thinking and discuss the conditions of the existence of solutions. On the other hand, memory of definitions and basic theorems, which could be understood as procedures, steps to follow or tests for categorization of objects, was most valuable. One could pass the LAI examination without engaging in proving activity. It was also better to be familiar with tacit notational conventions, established by repeated usage in the lectures, than expect rigorous mathematical formalism in the formulation of the questions and have doubts about the meaning of the variables used in them.

The LAII examination was more demanding in terms of systemic thinking (definitional, proving, hypothetical). This could account for the fact that less students achieved a high grade in LAII than in LAI. Another important reason was the sharp increase in the number of new concepts introduced in the LAII course combined with the fact that the final examination stressed remembering many definitions. LAI was largely a revision of a "Vectors and matrices" course that all students had to pass before entering LAI, and the only new concepts introduced were those related to linear transformations. LA II contained a much larger portion of new theory (inner product spaces and the theory of canonical decompositions of linear operators).

CHAPTER V. CONCLUSIONS

In this chapter we first reflect on our findings and their empirical value. We conclude by explaining why we think the cultivation of theoretical thinking is an important goal of university education.

A REFLECTION ON THE FINDINGS OF THE STUDY

The most striking feature of the behavior of students experiencing difficulties in linear algebra has long been, for us, their tendency to seek meaning in paradigmatic examples rather than in definitions of concepts. In the present research we looked at a group of "high achievers", that is, at students for whom, apparently, linear algebra was not too difficult. As expected, a common feature of their behavior was, what we have called, the definitional approach to meanings. At the same time, an analysis of the examination questions used in the linear algebra courses, in which these students achieved so well, suggested that this feature was highly relevant for high achievement. The assessment appeared to focus on the students' knowledge of a vast terminology, notation and definitions and not so much on relations between concepts, especially in the second course. The lectures developed a sophisticated structural theory, but the main achievements of the theory were absent from the assessment, which concentrated on a few technical details.

In this work we have expanded our notion of theoretical thinking beyond the definitional approach to meanings, and we have argued that a good understanding of linear algebra, in principle, requires thinking that is reflective, systemic and analytic. In this larger sense, theoretical thinking (and, therefore, understanding) turned out not to be necessary for high achievement in the courses taken by the students in our study. Not only understanding was not a necessary condition of high achievement, but being unable to temporarily suspend understanding while learning the "small steps" was such a traumatic experience for one of the students that she decided to give up mathematics altogether.

However, this conclusion must be moderated by the finding that theoretical thinking was much stronger in the subgroup of students who achieved highly in both courses than in the group of those who achieved highly in the first course.

In fact, it was stronger than necessary for high achievement. These students' potential did not meet with a challenge in the linear algebra courses.

As we analyzed the assessment items, we saw how little was needed, sometimes, to change a dull question into a challenging question that could open the way to a substantial theoretical inquiry. However, if we take seriously the assumption that theoretical thinking grows on the basis of and against practical thinking, i.e. that a theoretical thinker must be a practitioner of a theory, then more radical changes may be required, namely changes in the content of the courses. Why keep teaching a theory for which the undergraduate students have either no possibility or no opportunity to develop a practitioner's experience?

The teaching of abstract algebra (vector space theory, ring theory, group theory) to undergraduates has been introduced in the middle of the twentieth century, and also around that time it became possible to become a mathematician without having studied physics or any applications of mathematics. Linear algebra courses such as described in this work are remnants of the "new mathematics" curricular trends of the 60s and 70s. A few years ago, such courses came under a trenchant criticism of the Russian mathematician V. I. Arnold, in his talk in Paris during a debate on the teaching of mathematics:

> Mathematics is a part of physics. Physics is an experimental scienceÉand mathematics is the part of physics where experiments are very inexpensive.É In the middle of the twentieth century, an attempt was made to separate mathematics from physics. The results turned out to be catastrophic. Whole generations of mathematicians were growing up ignorant of half of their domain and having no idea of any other sciences. They started to teach their ugly scholastic pseudo-mathematics to students and then even to schoolchildrenÉ [The movement started in France] and it quickly spread to teaching basic mathematics in other countriesÉ Asked 'How much is 2 + 3?' the French primary school pupil would answer, '3 + 2, because addition is commutative'.ÉAnother French student gave this summary of mathematics: 'Here is a square but one still needs to prove it.'.É How could this happen in France which gave the world [such mathematicians] as Lagrange and Laplace, Cauchy and Poincar , Leray and Thom? É The attempt to create 'pure' deductive-axiomatic mathematics led to the rejection of the schema used in physics (observation, model, exploration of the model, conclusions, experimental testing of the conclusions) and to its replacement with the schema: definition, theorem, proof. It is impossible to understand a definition which is not motivated, but this doesn't stop the 'algebraists-axiomatists'..ÉWhat is a group? Algebraists teach that it is a set with an operation satisfying many easily forgotten axioms. Such a definition provokes a natural protest: why would any sensible person need such an operation?É The situation is completely different if we start not from the concept of group but from the concept of transformationÉjust as this concept developed historically. A set of [one-to-one] transformations is called a group if, together with any two transformations it contains their superposition and together with any transformation it

contains its inverse. This is a complete definition.É[t]here are no more 'abstract' groups in the worldÉ(my translation from Polish [12], AS).

We would not go as far as considering all of mathematics as a part of physics, but we certainly agree with the idea that teaching the structural theory of linear algebra without sensible motivation for the definitions and without applications does not make sense. Theoretical thinking without an underlying domain of practical experience which it attempts to question, organize, model, and understand better, is epistemologically futile and produces irrelevant knowledge.

In this research we looked at theoretical thinking as a possible factor in high achievement. Other researchers looked at different factors, such as epistemological positions, meta-cognitive strategies, persistence in study, affective attitudes, etc. It is difficult to say, what is the relative importance of these factors and how their role changes as a function of the content of the courses and methods of assessment. This could be an interesting question to study.

WHAT IS THE EMPIRICAL VALUE OF OUR FINDINGS?

In our research we collected some empirical data. We interviewed 14 students and we analyzed the examinations, based on which these students were considered "high achievers in linear algebra". The whole research project was inspired by empirical observations of students in difficulty. But the design of the research and the interpretation of the collected data were grounded in a theory, a theory of theoretical thinking.

What is then, we ask, the empirical value of our conclusions from this research? Couldn't perhaps these conclusions be derived analytically from the theory? Was the empirical research at all necessary?

This is an important question to ask. Empirical research is costly. If the conclusions of a research were analytic statements, logically derived from the assumptions of a theoretical framework, the collection of data would be a waste of time. Here is an example.

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[12] Excerpt from a talk given by V.I. Arnold on March 7, 1997, in le Palais de la D couverte in Paris, during a discussion on the teaching of mathematics. A Polish translation was printed in *Wiadomosci Matematyczne 2001*, 17-26.

In mathematics educators' discussions about students' difficulties in linear algebra, the blame has often been put on their poor logical thinking. This is an analytic statement, which follows from the definitions of linear algebra and logical thinking. Obviously, if one confuses quantifiers and negates them incorrectly then one is bound to have difficulties in, for example, the use of the definitions of linear dependence and independence in formal proofs. It was then proposed that students receive a special training in formal logic and proof methods prior to taking a linear algebra course. Empirical research was conducted to verify if, indeed, taking such courses would improve students' understanding of linear algebra. The outcome was to be expected: it did not (Dorier and Sierpinska, 2001). It did not because, logical thinking being a broader category than good understanding of linear algebra, the converse of the statement, "logical thinking is a necessary condition of a good understanding of linear algebra" does not hold. It was not necessary to conduct empirical research to reject this hypothesis. For similar logical reasons, we are not recommending, in our conclusions, that students should take critical thinking courses prior to the linear algebra courses at the university. This recommendation does not follow from our research: We cannot claim that taking such courses will necessarily improve their understanding of linear algebra. They may become more theoretically inclined in their thinking in general, but this intellectual disposition alone is not a sufficient condition for a good understanding of linear algebra. Theoretical thinking, in our model, is a necessary, but not a sufficient condition of a good understanding of linear algebra.

As mentioned, our research started with an empirical observation. We noticed a repeated co-occurrence of, on the one hand, contradictions with the linear algebra theory in students' mathematical statements and, on the other, a behavior that we later termed "practical" (e.g. constructing meaning of a concept on the basis of examples and treating definitions as descriptions of some properties of the concept). This behavior appeared to us as quite common and effective in everyday life and communication. This is why we called it "practical". Inspired by certain existing epistemological and psychological distinctions, we constructed a model of "theoretical thinking", embedded in a set of assumptions about cognition and learning. Within the frame of this theoretical framework, we could now make and analytically justify the statement that "theoretical thinking is a necessary condition of a 'good' understanding of linear algebra" (i.e. understanding free from contradictions with the basic concepts of this theory). This statement logically implied that "practical thinking yields poor understanding of linear algebra", thus turning our initial empirical observation into an analytic statement. However, without this empirical observation, we would have not had a motivation for developing a model of theoretical thinking.

In our research, we collected data about some students' grades in the linear algebra courses, we interviewed those who achieved a high grade in the first of these courses, and we "marked" their responses using a set of indices, which we called "expectation", "tendency" and "capability". We compared the values of these indices with the students' grades.

But what was the empirical value of the grades and the indices? One could perhaps say that the grades were less theory-laden than the indices. Surely, the grades did not depend on our model of theoretical thinking. But, in our research, grades were involved in our definition of "high achiever"; a high achiever being a student whose grade in a course was ‡ 80%. The indices depended on the assumed model of theoretical thinking. They were calculated according to certain formulas. We described the meaning of these indices in words, too. For example, the index witt% for a given student represented this students' *tendency* to behave theoretically (according to our definition of theoretical behavior) on a question in our interview. The resulting number represented, for us, our intuition of tendency. However, the number itself was defined not by this "description" but by the formula. Therefore, there would be no point in complaining that witt% does not fit someone else's understanding of the word "tendency". In our theory, "tendency" was a technical term.

The "expectation", "tendency" and "capability" indices depended on our model of theoretical thinking and its corresponding operational model of theoretical behavior. But the values of these indices depended a lot on how we interpreted the students' behavior in the interviews. Moreover, the definitions of the indices were a function of the way we decided to score the students' utterances. In particular, we chose not to distinguish between a single occurrence of a certain kind of behavior in an interview question, and multiple occurrences of this type of behavior. We could have settled on counting these occurrences and using these numbers as weights in our definitions of the indices. But we gave up this idea for practical reasons. It would have been very difficult, technically, to decide if a given "noise" that a student was making, whether a "yes", or a "no", an "mmm", a snort or a sigh, should count as representing a given kind of behavior on a par with a full phrase or a longer speech act. We did not see this level of technical detail as relevant for our research. Therefore, perhaps our conclusions could count as "synthetic judgments" (in Kantian sense), except that their reference would not be students' thinking but the researchers' thinking about the data.

Certainly, we cannot claim that the Question-by-question and Feature-by-feature tables "faithfully" represent the group's "actual" disposition to theoretical thinking, even in the sense of our assumed model. We don't think it is possible to say what the actual disposition was or could be.

But, if one insists on "facts", is there anything in our report that could count as being "as close as one can get" to a fact (assuming that there are such things as facts in the world of human cognition)? Surely, one could say that the final examination texts are a fact; they are documents of certain events. The protocols from the interviews are not far from a fact, but they are already further from it than the audiotapes and even further than the experience of sitting there and talking with the students. Verbatim citations from the protocols, taken out of the context of their occurrence, carry us even further away from this "lived reality". Still more distant are conclusions such as "All students in the group used some form of contradiction to refute a general statement", or "No one in the group engaged in axiomatic reasoning", because the meaning of these conclusions depends on the assumed meanings of the terms used in them (e.g. "contradiction", "axiomatic reasoning"). Closer to the fact than these conclusions are descriptions of the students' responses to the Brillig numbers question, "Is the sum of two brillig numbers a brillig number?", or the Vorpal question, "Is it possible to have an operation with distinct left- and right-hand zero elements?"

Perhaps the most distant from fact would be conclusions such as "Assessment in linear algebra courses does not discriminate between high and low disposition to theoretical thinking in students". This conclusion could be supported by the example of the student N1, who achieved high grades in both linear algebra courses but scored low on theoretical thinking, and by an analysis of the final examinations. But the conclusion and its justifications depended heavily on our model of theoretical thinking, our interpretations of the students' behavior in the particular interview. Moreover, the formulation of the conclusion sounds as a generalization to all courses and all students of what could be a very particular case. The student N1's high achievement and his low theoretical thinking scores could have been quite accidental, results of a random convergence of particular circumstances. Also the choice of the examination questions was the result of a bias (and whim) of a particular university professor at a particular time of his life.

What can we conclude?

Did it make sense to develop all this methodology for an assessment of the students' theoretical thinking? Did it make our conclusions more "objective"?

Anything we say about our observations of students' mathematical behavior is imbued with our personal interpretation, prejudice, theoretical preferences, and the like. But, as conscientious researchers, we try to be at least consistent in our interpretations; for example, we take care to apply the same criteria in evaluating the behavior of all the subjects in the study. It is easy to become favorably or unfavorably biased by the students' gentle or aggressive behavior, their calm or nervousness, silence or volubility. In our research we tried to harness our "impressions", and tame our fantasy in interpretation, by developing definitions of what would be considered as theoretical behavior. These definitions were not established prior to reading and analyzing the protocols. Tentative definitions were created during the first reading of the protocols. They were then tested through further readings. A first student's behavior in a question of the interview would be described and categorized as theoretical or not in these or other aspects. The second student's behavior in the same question would be analyzed already in comparison to the behavior of the first one. As a result the notion of theoretical behavior in this question would be refined, formulated using more precise terms. And this process would continue with the next student, until, at the end, we would have a first definition of theoretical behavior in this question. This definition would be tested on all the students again; the protocols would be read again with this new definition in mind. Often the definition would be changed and a third or even fourth reading would be necessary. Some students would display more features or different features of thinking than others in a given question. Then we would take the intersection of the sets of features of theoretical thinking displayed by the students, because we were interested in looking at the students as a group, and we wanted to talk about the disposition to theoretical thinking of the group of high achievers as a whole. The quantitative part of our interpretation was providing us with further means of control over our interpretations. An unexpected value of an index would make us go back to the protocols to see if this was just a mistake in our assignment of scores or if the student or students indeed behaved in a certain way.

Thus, the methodology was imposing a discipline on our analyses. In return, it allowed us to notice things to which we wouldn't have paid attention otherwise. We believe that it is thanks to this discipline that this report is not just a story about our research and not just an account of our impressions of students' mathematical behavior.

Why so much fuss about the development of theoretical thinking in our students? — you may ask. Because this research has been about teaching and learning mathematics in a *university*, and we tend to believe that development of theoretical thinking is one of the main goals of any university. This is what distinguishes this institution from a church, a party's militant cell, or a carpenter's workshop. The mission of a university is not to initiate students into a dogma, nor to prepare them for concrete jobs. A university serving the interests of a particular political party, religion or profession would not be worthy of this name. Therefore, it must be based on thinking, which distances itself from any particular ideology, religion, or practice, and which, without taking sides, looks at these cultural phenomena critically, comparatively, and historically, bringing to the fore their implicit assumptions, and all that they take for granted. A university is not based on plans of action, like political institutions, industry or business. Its work consists in the creation and validation of systems of concepts, which allow one to critically analyze the existing standard procedures or technical know-how. These systems are sometimes a basis for finding shortcuts and opening the way to technological innovations, but the latter are a by-product, not a goal in itself, of the work of a university. Rather than engaging in action, a university takes the time necessary to reason out the consequences of the possible courses of action. Universities today use ethnic languages (not Latin) in their courses. But language at a university is always regarded as one of possible languages, just as any theory is regarded as one of the possible theories. University professors often publish their papers in a language different from the language of their instruction and participate in international conferences where they have to understand representatives of other universities and be understood by them. Thus, they must distance themselves from the language, and have their thinking control the language rather than let the patterns and biases of one ethnic language control their thinking.

Education of the Intellectual

The "idea" or the "mission" of a university has been the object of many discussions and philosophical discourses, taking place in different political contexts and stressing various aspects of knowledge as the goal of university education. For example, John Henry cardinal Newman's discourses, delivered to the Catholics of Dublin in 1852 and later published under the title of "The idea of a university", were situated in the context of debates about the secularization of education. Newman was reflecting on the differences among Useful, Liberal and Religious kinds of knowledge and proposed that an ideal university should be the committed to Liberal Knowledge (Newman, 1889, p. 121-123). Newman was stressing the self-serving character of theoretical knowledge, subordinating to this feature all other features. Newman’s vision of theoretical knowledge could appear to us today as a severe, strict discipline of the mind, with little space for originality and creativity, something like the formal and smooth beauty of the classical paintings of Ingres.

Education of the Critical Thinker

More than a hundred years later, in 1967, J rgen Habermas was shifting the focus towards flexible and critical thinking. In a way, he contested Newman’s idea of the "cultivation of intellectual excellence" as the main goal of university education. His discussion of the idea of a university was taking place at a time when, on the one hand, some states were investing in the development of technological and other professional higher education institutions, and, on the other, the traditional German university, focused on purely theoretical knowledge was the target of severe criticisms (and, indeed, boisterous student protests). He agreed that "universities must transmit technically exploitable knowledgeÉand also produce it", but he stressed that an important task of a university is "to form the political consciousness of its students" (Habermas, 1971, p. 2-3). This task, he said, was very much neglected by the traditional German university, focused on classical philosophy, and "ideal" theoretical knowledge. Apolitical education led to political conformism or a mentality of "loyalty to state authority" which had well-known political consequences. Although the post-war ideological climate encouraged political education of the students, this education was confined to taking courses in Political Science departments. This extension of the curriculum did not change the traditional focus of the German university on "self-understanding" (ibid, p. 5), or what we have called the self-serving character of theoretical knowledge. Habermas was proposing a way to reforming the German university by opening it up to practical reason while not giving up the development of theoretical thinking. He saw a "subterranean unity of theoretical and practical reason" (p. 7) in what he called "critical thinking". Habermas’ notion of critical thinking seems to us no different from our notion of theoretical thinking, because we have included hypothetical thinking and meta-theoretical reflection in our definition. But Habermas used the terms of "theoretical" and "practical" in the sense of the logical positivistic interpretations of Hume’s distinction. He did not agree with these interpretations, and his notion of "critical" thinking was an attempt at overcoming this narrow understanding of the terms. For him, the logical separation between the descriptive statements of theoretical knowledge and the rules of communicative action of practical knowledge did not imply an institutional separation. Indeed, he stressed that the processes of theoretical thinking always include judgment or meta-theoretical reflection about such things as the utility or effectiveness of certain research instruments and frameworks. This could be regarded as "practical thinking" but this thinking is, in fact, part and parcel of the process of the scientific inquiry (ibid, p. 6). He wrote,

> Now the argument propounded by Hume is not false. But I believe that it does not imply the strategy for which Hume’s positivistic successors have invoked it. We do not need to judge scientific inquiry only under the logical conditions of the theories that it generates. For another picture emerges if we examine not the results of the process of inquiry but its movement. Thus meta-theoretical discussions are the medium of scientific progress — I mean the utility of an analytic framework, the expedience of research strategies, the fruitfulness of hypotheses, the choice of methods of investigation, the interpretation of the results of measurement, and the implicit assumptions of operational definitions, not to mention discussion of theoretical foundations of the fruitfulness of different methodological approaches.
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> (Habermas, 1971, p. 6-7)

Education of the Professional

While German intellectuals condemned the excess of narrowly understood theoretical thinking in their universities, in North America criticism of universities was often directed against the overindulgence in practical thinking. Some authors didn’t mince words in their arguments. Hutchins (1936), for example, used the metaphor of "service-station" in describing the American idea of a university.

> Even more important is the influence on educational policy of student fees.É [M]ost of the things that degrade [universities] are done to maintain or increase this income. To maintain or increase it the passing whims of the public receive the same attention as those of millionaires. If the public becomes interested in the metropolitan newspaper, schools of journalism instantly arise.ÉDuring the synthetic excitement of last year about communism, socialism, and other forms of redness, it suddenly became the duty of the colleges and universities to give courses in the eradication of these great evils and in the substitution for them of something called Americanism.
>
> Undoubtedly the love of money and that sensitivity to public demands that it creates has a good deal to do with the *service-station* [our emphasis] conception of a university. According to this conception a university must make itself felt in the community; it must

> be constantly, currently felt. A state university must help the farmers look after their cows. An endowed university must help adults get better jobs by giving them courses in the afternoon and evening.
>
> (Hutchins, 1936, p. 5-6)

Of course, this is a caricature, and Hutchins meant it to be a caricature. Not all institutions of higher education in America followed this pattern. But those that did not were few and they were highly elitist. Students who came to study in these institutions were already graduates of selected secondary schools or followed special programs in secondary schools, where the focus was on literary dissertations, logical reasoning, mathematics and science. Resnick (1987) called this view of education, the "high literacy" tradition. For her, this tradition promoted "higher order thinking", i.e. thinking that is at least nonalgorithmic, complex, yielding multiple solutions, based on nuanced judgment and interpretation, involving the effort of finding structure in apparent disorder, while trying to deal with the inevitable uncertainty in this mental work using a self-regulation of the thinking process (ibid, p. 3). She wrote,

> In America, various 'academies', some private and some public, carried on this tradition [of "higher literacy"] through the nineteenth century and into the twentieth. Until they began to be transformed early in this century [20th], even public high schools were in the academy mold. Only a minority of young people attended, or even thought of attending them. There were entrance examinations. The curriculum was strictly academic. Extensive writing, textual criticism, and the like were expected. Although today we might not recognize nineteenth-century academy curricula as promoting creative thinking or independent problem-solving, the elite academies expected to produce, and to a considerable extent succeeded in producing, intellectual performance beyond the ordinary.
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> (Resnick, 1987, p. 4).

Hutchins' point was not to replace the curricula of the non-elitist state and endowed American universities by the purely "academic" content as outlined in the citation above. He did not propose that American universities get rid of professional schools altogether; he only advised not to trade "professional emphasis" for "vocational emphasis" (ibid, p. 54). "From the university standpoint Ñhe stressed Ñat least, a professional discipline to be a professional discipline must have intellectual content, and have it in its own right" (ibid.). Hutchins' focus was on what we have named the reflective character of theoretical thinking.

> If the universities can revert to a condition where the number of professional schools and courses is limited to those that have intellectual content in their own right, they will have gone some distance toward disposing of the dilemma of professionalism. They will go still farther toward disposing of it if they can insist that the professional schools and

> departments that remain deal with their subject matter in the true university spirit, that is, in the spirit of studying them for their own sake. Every learned profession has a great cultural heritage, and it is this, which should be the prime object of attention of professional schools. I believe that these schools will find that their students will be better prepared for practice if they are trained to think in the subject matter of the professional discipline than if they have been taught by the cookbook method.
>
> (Hutchins, 1936, pp. 56-57).

Hutchins' point was that, while vocational training without theoretical thinking may help the graduates get jobs; it does not guarantee that they will be able to advance in the profession or even keep the jobs. For a student in an actuarial mathematics program, it would be important to know that economical models that used to work well, may no longer apply in some new conditions. It is theoretical thinking that allows one to adapt to the changing conditions thanks to an understanding of the models used in the profession, seeing how they depend on certain assumptions and being able to verify to what extent these assumptions are satisfied. Theoretical thinking allows *improvisation* with mathematical models (King, 2001), something, without which one cannot survive and make progress as an applied mathematician. Paradoxically, theoretical thinking is the most practical gift we can offer our students.

Education of the Intellectual Actuary

The context of our research on theoretical thinking in high achievers was that of undergraduate linear algebra courses for mathematics specialization and actuarial mathematics students, situated within a Mathematics & Statistics department of an English-Canadian University in Montreal, with many of its students coming from a French-Canadian background, and many from recent immigrants' families. Lectures are in English, but students speak other languages among themselves. After a pre-university education in French, whether by choice or by law[13], many come to the English university to increase their chances of getting a job in the mostly English speaking North America. The actuarial program in an English university further increases their chances of getting a job. But there are no special mathematics classes for actuarial students in our department; these students are sitting together with students aiming at going on to do an M.Sc. or even a Ph.D. in mathematics. There are not enough "pure mathematics" students at the university to open special classes for them.

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[13]"Law 101" in Qu bec obliges immigrants' children to follow elementary and secondary education in French, unless either the parents' pre-college education was in English, or the parents can afford sending

The lecturers of the mathematics courses are, therefore, deeply caught in the "dilemma of professionalism". In the department, this dilemma is disposed of (we hesitate to say, "solved") by a separation of theoretical and practical thinking. The separation is obtained along two axes: the separation of theory and applications in the content of courses, and the division of responsibilities for different kinds of knowledge between teachers and students. Courses are divided into "pure mathematics" courses (labeled "MATH") and "applied mathematics" courses for the actuarial profession (labeled "ACTU"). In any course, the teacher takes care of the theory; the student is responsible for remembering certain definitions and procedures. The needs of students with an inclination for theoretical thinking are satisfied by offering them individual attention of the faculty. These students may take extra reading courses and participate in undergraduate projects.

In the regular undergraduate mathematics courses, "theoretical thinking" is understood more in the logical positivistic sense, than in the sense of "critical thinking" as proposed by Habermas. Indeed, "critical thinking" in North American colleges and universities has received a very special meaning. It is the name of a course that a student takes in college, gets a grade, and is thus expected to have acquired the intellectual disposition of an "educated person" for life.

One might think that mathematics courses offer an ideal environment for fostering critical thinking in students, with the distinctly hypothetical character of this knowledge and its notorious concern with validation of its statements. Unfortunately, as we all know very well and our research only confirms it, it is very easy to ruin this opportunity. We teach students certain schemes of thought, but we often stop short of awarding them the license for these schemes of thought. They master them, in the best of cases, as technical knowledge, they do not construct these schemes as theoretical knowledge, which would give them the feeling of ownership of this knowledge, freedom to change it, apply and develop further.

Our analysis of the final examinations in the two linear algebra courses has shown how little of the theory introduced in the course the students needed to achieve a high grade. The theory was all about making conversions from one language to another; from the language of linear equations to the language of matrices, from the language of linear transformations to the language of matrices, from one matrix representation to another of a

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their children to a private school, in which case they can choose an English or a French language school.

linear operator; representing diagonalizable operators as linear combinations of especially simple mappings, representing linear operators in the convenient Jordan Canonical form, diagonalizing quadratic forms. But, for example, for all the effort put into proving the possibility and conditions of the Jordan canonical representations, the students were not given the chance of *using* such representations as means to solve other problems. They were not given a chance to do what the theory was aimed at doing. They were left with the exercises of the type: "Find all possible Jordan canonical forms for those matrices whose characteristic and minimal polynomials are as followsÉ. *The theory remained the frolic of the university professor*.

Of course, one could say, students had to write some proofs, so they had some control over the theory. But, from the interviews, it appeared that *students considered proofs as a special kind of mathematical exercises,* characteristic of, especially, linear algebra courses. They were not very likely to conceive of proof as a way of validating a mathematical statement, and therefore as means of control over their solutions.

Education of the Persuasive Spokesperson

The feature of theoretical thinking that the students in our study seemed to be most inclined to, was definitional thinking. Definitional thinking, without the distancing afforded by reflective and hypothetical thinking, is a very common instrument in rhetoric. Identifying theoretical thinking with definitional argumentation, some may argue that a university focused on the development of theoretical thinking would serve only to produce skilled political speakers, apt at justifying unpopular governmental decisions or even unethical behavior.

Of course, theoretical argumentation can be used for various political reasons. But, *by definition*, theoretical argumentation cannot justify behavior, any behavior, and not only unethical behavior. Theoretical argumentation may only validate a theoretical statement, using other theoretical statements and following rules of validation conventionally admitted in a theory. And theoretical thinking always conceives this argumentation as valid under the constraints of this particular theory, which is not the only possible theory. Theoretical thinking, by virtue of its hypothetical and analytic properties, contains a meta-level reflection, which makes it aware of the conditional status of the statements it makes. Theoretical thinking is not reduced to definitional thinking within the frame of a single theory. If a theory is regarded as the one and the only valid theory of something by the thinker, it starts to function as an "ideology", a dogma.

By privileging the learning of a terminology and definitional thinking over other features of theoretical thinking in our mathematics courses, we are not helping students to distinguish between theoretical and dogmatic thinking, or even between theoretical argumentation and persuasive rhetoric, for that matter.

Education of the Conscientious Clerk

The system of assessment through written tests, common for all sections of a course, does not encourage investigative, research-like attitudes in students, posing and verifying conjectures and asking "what-if" questions. The very context of a written standard examination rules out the attitude of "studying for the sake of studying". The goal is to achieve a good grade and to succeed socially/academically, not to further one's knowledge of the domain. It is not in the interest of the student to do more than required in a given problem, because time is limited, and all problems must be solved. The result is that the inclination and ability to pose interesting conjectures does not discriminate between high and low achievers. In our interviews both O2 and O3 surprised us with puzzling mathematical hypotheses, but O2 failed the course and O3 achieved an A+.

Peer Education of the Theoretical Thinker

It may well be that the mission of the university is not accomplished in its lecture halls and compulsory courses. The students in our study made references to their vast readings and reflection and to conversations with their friends on a variety of philosophical and mathematical topics. At least two of them had a book with a popular account of the discovery of the proof of Fermat's Last Theorem in their backpacks.

Students follow courses and pass examinations, but, for the best of them, this may be just a small part of their university education. As Andr Joyal, a mathematician at Universit de Montr al put it in an interview [14], "*I studied on my own and I became a kind of self-taught man, even if I had [to take] some courses at the university. In fact, I was rather absent in classes*É*In general, I was not going to classes, except for taking the examinations. But I*

[14] http://perso.wanadoo.fr/jacques.nimier/entretien

was present at the university, that is, I used to work with some students who were as enthusiastic as I was for mathematics and we used to work together, we were asking ourselves all kinds of questions, and this way I also learned a lot. The courses, it was for the examinations, something formal, because, after all, one had to earn a diploma." (My translation, AS)

So maybe there is no reason for lamenting over the low theoretical component of the linear algebra examinations. These examinations are but formalities, which provide the necessary documentation for the university administration to decide who will receive a university diploma. Perhaps the mathematics professors simply have to accept it as a fact that they play only a very small part in the education of the future theoretically thinking professional.

There is a difference, however, between this kind of modesty, and honesty. It is all right to contribute only a *little* to the development of students' theoretical thinking. But we don't think it is all right to accept that students achieve high grades in a course without understanding its main ideas. If we don't think that students are able, at a given stage of their studies, to understand the main ideas of a theory, then courses about this theory should be postponed till later in their studies, or the theory should be taught only in some specialized programs of study. We thus come back to the question we posed at the beginning of this work: if linear algebra cannot be understood well enough without theoretical thinking, and theoretical thinking is not necessary for success in a given linear algebra course, then what is the point of making this course compulsory for so many students?

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APPENDIX I

QUESTION-BY-QUESTION TABLES

QUESTION-BY-QUESTION TABLE

	Question 1. "Classification"	°	°	°	°		°	°	°
	TB21a (form.categ)		TB311a (rigour)		TB321a (variables)		iEq%	ittq%	iCq%
O1	0	1	0	1	0	1	0.00	0.00	0.00
O2	0	1	1	0	0	1	33.33	33.33	33.33
O3	1	0	1	0	1	0	100.00	100.00	100.00
O4	1	0	0	1	1	0	66.67	66.67	66.67
V1	1	1	0	1	1	0	50.00	50.00	66.67
V2	0	1	1	0	1	0	66.67	66.67	66.67
V3	1	0	0	1	1	0	66.67	66.67	66.67
V4	1	0	1	0	1	0	100.00	100.00	100.00
S1	1	1	0	1	0	1	16.67	25.00	33.33
S2	1	0	0	1	1	0	66.67	66.67	66.67
S3	1	0	1	0	0	1	66.67	66.67	66.67
S4	1	0	1	0	1	0	100.00	100.00	100.00
N1	1	1	0	1	1	0	50.00	50.00	66.67
N2	1	0	0	1	1	0	66.67	66.67	66.67

Question 2. "Linear independence definition"				°	°	°	°
°	TB321a (variables)		TB23a (hypothetical)		iEq%	ittq%	iCq%
O1	0	1	0	1	0	0	0
O2	0	1	0	1	0	0	0
O3	1	0	1	0	100	100	100
O4	1	0	0	1	50	50	50
V1	1	0	1	0	100	100	100
V2	1	0	0	1	50	50	50
V3	1	0	1	0	100	100	100
V4	1	0	0	1	50	50	50
S1	0	1	1	0	50	50	50
S2	1	0	0	1	50	50	50
S3	0	1	0	1	0	0	0
S4	1	0	1	0	100	100	100
N1	1	0	1	0	100	100	100
N2	1	0	1	0	100	100	100

Question 3. "Linear dependence typo"	TB21b (def)		TB2a (prv)		TB23a (hyp)		TB312a (termin)		TB322a (quant)		TB322b (connectives)		iEq%	it
O1	1	0	0	1	0	1	1	0	1	0	1	0	66.67	66
O2	0	1	0	1	0	1	0	1	0	1	0	1	0.00	0
O3	1	0	1	0	1	0	1	0	1	0	1	0	100.00	10
O4	1	1	0	1	1	1	0	1	0	1	0	1	16.67	25
V1	1	0	1	1	1	1	1	0	0	1	0	1	50.00	50
V2	1	0	1	1	1	1	1	0	1	0	1	0	83.33	75
V3	1	0	1	1	1	1	0	1	1	0	1	0	66.67	62
V4	1	1	0	1	0	1	0	1	0	1	0	1	8.33	14
S1	1	0	1	0	1	0	1	0	1	0	1	0	100.00	10
S2	1	0	1	1	1	1	0	1	1	0	1	0	66.67	62
S3	1	0	1	0	1	0	0	1	1	0	1	0	83.33	83
S4	1	0	1	0	1	0	1	0	1	0	1	0	100.00	10
N1	1	0	0	1	0	1	0	1	1	0	1	0	50.00	50
N2	1	0	1	1	1	1	0	1	1	0	1	0	66.67	62

Question 4. "Log-log scales"

	TB1a (res)		TB21c (def-graf)		TB321b (graf-rel)		iEq%	ittq%
O1	0	1	1	1	1	1	33.33	40.00
O2	1	0	1	1	1	1	66.67	60.00
O3	1	0	1	1	1	1	66.67	60.00
O4	0	1	1	1	1	1	33.33	40.00
V1	0	1	1	0	1	0	66.67	66.67
V2	1	0	1	0	1	0	100.00	100.00
V3	1	0	1	0	1	0	100.00	100.00
V4	0	1	1	1	1	1	33.33	40.00
S1	0	1	1	0	1	0	66.67	66.67
S2	0	1	1	1	1	0	50.00	50.00
S3	0	1	1	0	1	1	50.00	50.00
S4	0	1	1	0	1	1	50.00	50.00
N1	0	1	1	1	1	0	50.00	50.00
N2	0	1	1	0	1	0	66.67	66.67

Question 5. "Brillig numbers"										
	TB1b (intr sig)		TB322c (form def)		TB322d (implic)		TB22b (refutation)		iEq%	ittq
O1	1	1	1	0	1	0	1	0	87.50	80.(
O2	1	0	0	1	1	1	1	0	62.50	60.(
O3	1	1	1	0	1	0	1	0	87.50	80.(
O4	1	0	0	1	1	1	1	0	62.50	60.(
V1	1	0	1	0	1	1	1	0	87.50	80.(
V2	1	0	1	0	1	0	1	0	100.00	100.
V3	1	0	1	0	1	1	1	0	87.50	80.(
V4	1	0	1	0	1	1	1	0	87.50	80.(
S1	1	1	1	0	1	0	1	0	87.50	80.(
S2	1	1	0	1	1	1	1	0	50.00	50.(
S3	1	0	1	0	1	1	1	0	87.50	80.(
S4	1	0	1	1	1	1	1	0	75.00	66.6
N1	1	1	0	1	0	1	1	0	37.50	40.(
N2	1	0	1	0	1	1	1	0	87.50	80.(

Question 6. "Vorpal"	TB21b (def)		TB22a (prov)		TB22c (axiom. reas)		TB322a (quantifiers)		iEq%	ittq
O1	1	1	0	1	0	1	1	1	25.00	33.
O2	1	0	1	0	0	1	1	1	62.50	60.
O3	1	0	1	0	0	1	1	0	75.00	75.
O4	1	0	0	1	0	1	1	0	50.00	50.
V1	1	0	1	1	0	1	1	0	62.50	60.
V2	1	0	1	0	0	1	1	1	62.50	60.
V3	1	0	0	1	0	1	1	1	37.50	40.
V4	1	0	0	1	0	1	1	0	50.00	50.
S1	1	0	1	1	0	1	1	0	62.50	60.
S2	1	0	0	1	0	1	1	1	37.50	40.
S3	1	0	0	1	0	1	1	0	50.00	50.
S4	1	0	0	1	0	1	1	1	37.50	40.
N1	1	0	1	1	0	1	1	1	50.00	50.
N2	1	0	1	1	0	1	1	0	62.50	60.

Question 7. "Beliefs"

	TB1c (relational)		TB1a (researcher’s)		TB23a (hypothetical)		iEq%	ittq%
O1	0	1	0	1	0	1	0.00	0.00
O2	0	1	1	1	1	1	33.33	40.00
O3	1	0	1	0	1	0	100.00	100.00
O4	1	1	1	0	1	1	66.67	60.00
V1	1	0	1	1	1	1	66.67	60.00
V2	1	1	1	0	1	1	66.67	60.00
V3	1	1	1	0	1	0	83.33	75.00
V4	1	0	1	0	1	1	83.33	75.00
S1	1	0	1	0	1	0	100.00	100.00
S2	1	0	0	1	0	1	33.33	33.33
S3	1	1	1	1	1	1	50.00	50.00
S4	1	0	0	1	0	1	33.33	33.33
N1	1	0	0	1	0	1	33.33	33.33
N2	1	0	1	1	1	1	66.67	60.00

Question-by-question table

WEIGHTED AVERAGES	°			GRADES °	
wiE%	witt%	wiC%		LA I	LA II
38.00	38.93	48.00	O1	**92**	70
36.00	35.20	48.00	O2	**82**	50
90.00	88.00	96.00	**O3**	**83**	**90**
46.00	47.60	60.00	O4	**82**	63
66.00	63.60	80.00	V1	**82**	70
78.00	74.80	88.00	**V2**	**98**	**85**
74.00	71.20	84.00	**V3**	**82**	**80**
54.00	54.03	64.00	V4	**93**	60
74.00	73.40	80.00	S1	**85**	°
52.00	51.40	64.00	S2	**88**	°
62.00	60.80	72.00	**S3**	**92**	**85**
72.00	71.07	80.00	S4	**88**	77
50.00	50.40	60.00	**N1**	**100**	**90**
72.00	68.60	84.00	**N2**	**100**	**85**
			cor Gr/wiE%	-0.01	**0.69**
			cor Gr/witt%	-0.02	**0.7**
			cor Gr/wiC%	-0.02	**0.66**

APPENDIX II. FEATURE-BY-FEATURE TABLES

Reflective thinking

Students	1 TB1a (res)	Q.4,7	2 TB1b (signif)	Q.5	3 TB1c (rel)	Q.7
O1	0	1	1	1	0	1
O2	1	1	1	0	0	1
O3	1	0	1	1	1	0
O4	1	1	1	0	1	1
V1	1	1	1	0	1	0
V2	1	0	1	0	1	1
V3	1	0	1	0	1	1
V4	1	1	1	0	1	0
S1	1	1	1	1	1	0
S2	0	1	1	1	1	0
S3	1	1	1	0	1	1
S4	0	1	1	0	1	0
N1	0	1	1	1	1	0
N2	1	1	1	0	1	0
E=(2t+h)/28		0.46		0.82		0.71
gtt=(t+h)/(14+h)		0.48		0.74		0.67
C=(t+h)/14		0.71		1.00		0.86

Systemic thinking, continued

Students	4 TB21a (cat)	Q.1	5 TB21b(def-a)	Q.3,6	6 TB21c(def-g)	Q.4
O1	0	1	1	1	1	1
O2	0	1	1	1	1	1
O3	1	0	1	0	1	1
O4	1	0	1	1	1	1
V1	1	1	1	0	1	0
V2	0	1	1	0	1	0
V3	1	0	1	0	1	0
V4	1	0	1	1	1	1
S1	1	1	1	0	1	0
S2	1	0	1	0	1	1
S3	1	0	1	0	1	0
S4	1	0	1	0	1	0
N1	1	1	1	0	1	1
N2	1	0	1	0	1	0
E=(2t+h)/28		0.68		0.86		0.75
gtt=(t+h)/(14+h)		0.65		0.78		0.67
C=(t+h)/14		0.79		1.00		1.00

Systemic thinking, continued

Students	7 TB22a (prv)	Q.3,6	8 TB22b (refut)	Q. 5	9 TB22c (ax-r)	Q.6
O1	0	1	1	0	0	1
O2	1	1	1	0	0	1
O3	1	0	1	0	0	1
O4	0	1	1	0	0	1
V1	1	1	1	0	0	1
V2	1	1	1	0	0	1
V3	1	1	1	0	0	1
V4	0	1	1	0	0	1
S1	1	1	1	0	0	1
S2	1	1	1	0	0	1
S3	1	1	1	0	0	1
S4	1	1	1	0	0	1
N1	1	1	1	0	0	1
N2	1	1	1	0	0	1
E=(2t+h)/28		0.43		1.00		0.00
gtt=(t+h)/(14+h)		0.46		1.00		0.00
C=(t+h)/14		0.79		1.00		0.00

Systemic thinking, continued

Students	10 TB23a (hyp)	Q.2,3,7
O1	0	1
O2	1	1
O3	1	0
O4	1	1
V1	1	1
V2	1	1
V3	1	1
V4	1	1
S1	1	0
S2	1	1
S3	1	1
S4	1	1
N1	1	1
N2	1	1
E=(2t+h)/28		0.54
gtt=(t+h)/(14+h)		0.52
C=(t+h)/14		0.93

Analytic thinking

	11		12		13	
Students	TB311a(rig)	Q.1	TB312a(term)	Q.3	TB321a(var)	Q.1,2
O1	0	1	1	0	0	1
O2	1	0	0	1	0	1
O3	1	0	1	0	1	0
O4	0	1	0	1	1	0
V1	0	1	1	0	1	0
V2	1	0	1	0	1	0
V3	0	1	0	1	1	0
V4	1	0	0	1	1	0
S1	0	1	1	0	0	1
S2	0	1	0	1	1	0
S3	1	0	0	1	0	1
S4	1	0	1	0	1	0
N1	0	1	0	1	1	0
N2	0	1	0	1	1	0
E=(2t+h)/28		0.43		0.43		0.71
gtt=(t+h)/(14+h)		0.43		0.43		0.71
C=(t+h)/14		0.43		0.43		0.71

Analytic thinking, continued

	14		15		16	
Students	TB321b(grf)	Q.4	TB322a(qua)	Q.3,6	TB322b(con)	Q.3
O1	1	1	1	1	1	0
O2	1	1	1	1	0	1
O3	1	1	1	0	1	0
O4	1	1	1	1	0	1
V1	1	0	1	1	0	1
V2	1	0	1	1	1	0
V3	1	0	1	1	1	0
V4	1	1	1	1	0	1
S1	1	0	1	0	1	0
S2	1	0	1	1	1	0
S3	1	1	1	0	1	0
S4	1	1	1	1	1	0
N1	1	0	1	1	1	0
N2	1	0	1	0	1	0
E=(2t+h)/28		0.75		0.64		0.71
gtt=(t+h)/(14+h)		0.67		0.58		0.71
C=(t+h)/14		1.00		1.00		0.71

Analytic thinking, continued

Students	17 TB322c (fr-df)	Q.5	18 TB322d (im)	Q.5
O1	1	0	1	0
O2	0	1	1	1
O3	1	0	1	0
O4	0	1	1	1
V1	1	0	1	1
V2	1	0	1	0
V3	1	0	1	1
V4	1	0	1	1
S1	1	0	1	0
S2	0	1	1	1
S3	1	0	1	1
S4	1	1	1	1
N1	0	1	0	1
N2	1	0	1	1
E=(2t+h)/28		0.68		0.61
gtt=(t+h)/(14+h)		0.67		0.57
C=(t+h)/14		0.71		0.93

TOTALS	number of features = 18			
Sum(TT)	Sum(PT)	it=18-PT	ip=18-TT	ih=18-TT-PT
10	13	5	8	5
11	15	3	7	8
17	4	14	1	3
12	14	4	6	8
15	9	9	3	6
16	6	12	2	4
15	8	10	3	5
14	11	7	4	7
15	7	11	3	4
13	11	7	5	6
15	9	9	3	6
16	8	10	2	6
12	12	6	6	6
15	7	11	3	4

Feature-by-feature individual indices

iE=(2it+ih)/36*100				GRADES	
iE%	itt%	iC%		LA I	LA II
41.67	43.48	55.56	O1	**92**	**70**
38.89	42.31	61.11	O2	**82**	**50**
86.11	80.95	94.44	O3	**83**	**90**
44.44	46.15	66.67	O4	**82**	**63**
66.67	62.50	83.33	V1	**82**	**70**
77.78	72.73	88.89	V2	**98**	**85**
69.44	65.22	83.33	V3	**82**	**80**
58.33	56.00	77.78	V4	**93**	**60**
72.22	68.18	83.33	S1	**85**	
55.56	54.17	72.22	S2	**88**	
66.67	62.50	83.33	S3	**92**	**85**
72.22	66.67	88.89	S4	**88**	**77**
50.00	50.00	66.67	N1	**100**	**90**
72.22	68.18	83.33	N2	**100**	**85**
			cor Gr/iE%	0.06	**0.68**
			cor Gr/itt%	0.04	**0.68**
			cor Gr/iC%	-0.02	**0.57**

	Ranking according to E		Whole group
TB	**E**	gtt	C
22b (ref)	1	1	1
21b (def-a)	0.86	0.78	1
1b (sig)	0.82	0.74	1
21c (def-g)	0.75	0.67	1
321b (grph)	0.75	0.67	1
1c (rel)	0.72	0.67	0.86
321a (var)	0.71	0.71	0.71
322b (con)	0.71	0.71	0.71
21a (cat)	0.68	0.65	0.79
322c (f-def)	0.68	0.67	0.71
322a (quan)	0.64	0.58	1
322d (imp)	0.61	0.57	0.93
23a (hyp)	0.54	0.52	0.93
1a (res)	0.46	0.48	0.71
22a (prv)	0.43	0.46	0.79
311a (rig)	0.43	0.43	0.43
312a (term)	0.43	0.43	0.43
22c (ax-r)	0	0	0

	Ranking according to gtt		Whole group
TB	E	**gtt**	C
22b (ref)	1	1	1
21b (def-a)	0.86	0.78	1
1b (sig)	0.82	0.74	1
321a (var)	0.71	0.71	0.71
322b (con)	0.71	0.71	0.71
21c (def-g)	0.75	0.67	1
321b (grph)	0.75	0.67	1
1c (rel)	0.72	0.67	0.86
322c (f-def)	0.68	0.67	0.71
21a (cat)	0.68	0.65	0.79
322a (quan)	0.64	0.58	1
322d (imp)	0.61	0.57	0.93
23a (hyp)	0.54	0.52	0.93
1a (res)	0.46	0.48	0.71
22a (prv)	0.43	0.46	0.79
311a (rig)	0.43	0.43	0.43
312a (term)	0.43	0.43	0.43
22c (ax-r)	0	0	0

	Ranking according to C		Whole group
TB	E	gtt	**C**
22b (ref)	1	1	1
21b (def-a)	0.86	0.78	1
1b (sig)	0.82	0.74	1
21c (def-g)	0.75	0.67	1
321b (grph)	0.75	0.67	1
322a (quan)	0.64	0.58	1
322d (imp)	0.61	0.57	0.93
23a (hyp)	0.54	0.52	0.93
1c (rel)	0.72	0.67	0.86
21a (cat)	0.68	0.65	0.79
22a (prv)	0.43	0.46	0.79
321a (var)	0.71	0.71	0.71
322b (con)	0.71	0.71	0.71
322c (f-def)	0.68	0.67	0.71
1a (res)	0.46	0.48	0.71
311a (rig)	0.43	0.43	0.43
312a (term)	0.43	0.43	0.43
22c (ax-r)	0	0	0