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# Reasons of students' dependence on teachers and lack of sensitivity to mathematical contradictions

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- Witam Państwa w Warszawie !

"Rezygnacja z dążenia do prawdy jest przyznaniem się do porażki. Stwierdzenie zaś, że się prawdę posiada jest intelektualnym nadużyciem".

(To give up trying to find the truth is to accept defeat. But to claim that one possesses the truth is intellectual abuse.)

- Barbara Skarga, 2007, *Czlowiek to nie jest piękne zwierzę, Krakow: Wydawnictwo Znak.* (Humans are not beautiful animals) (p. 185)

# The context of this talk

- Research on sources of students' frustration in prerequisite mathematics courses *

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* Sierpinska, A., Bobos, G., Knipping, C.: Sources of students' frustration in pre-university level, prerequisite mathematics courses, to appear in *Instructional Science*.

# The context

- Questionnaire item

"*I need the teacher to tell me if I am right or wrong*"

  - Agree 67% out of 96 respondents
  - Disagree 22% out of 96 respondents

# The context

Yet it is believed
that mathematics develops
an independent, critical mind…

# The context

- Questionnaire item

Given a problem: Solve $|2x – 1| > 5$
Which solution do you like best? Why?

*Solution (a)*

$2x-1=5$ neg. $2x-1 = -5$
$x= 3$ $x = -2$
Answer: $-2 < x < 3$

Solution (b)

We use the theorem: $|a|>b \Leftrightarrow a < -b$ or $a > b$
$|2x-1|>5 \Leftrightarrow 2x-1<-5$ or $2x-1>5 \Leftrightarrow x<-2$ or $x>3$
Answer $x < -2$ or $x > 3$.

**62%** out of 96 respondents chose Solution (a)
20% out of 96 respondents chose Solution (b)

**Only 3 respondents noticed that Solution (a) is incorrect.**

# The context

Yet, it is said that one of the purposes
of mathematics teaching is
"**nurturing the sense for truth**"
because mathematics is
a "**science that begins from common sense
and pursues certainty**";
therefore "teachers must have the attitude of passionate **concern about truth**" and pass it on
to their students.

(according to Pestalozzi, Freudenthal and Peters)

# The context

- Interviews with students

  "I failed the prerequisite Calculus course the first time around…
  So I took it a second time in the summer term…
  My teacher in the summer,
  he was a great teacher
  he explained well and everything
  but it's just that I could never grasp,
  like **I could never be comfortable enough to sit down in front of an example and do it on my own**,
  instead of looking back at my notes….
  **I just never understood the logic behind it**…
  How I studied for the finals?
  I was looking at the past finals.
  **All by memorizing**, that's how I passed the second time."

(21 years old, female, taking Calculus courses as prerequisite for admission into Commerce)

# The context

Yet, it is said that one of the purposes of mathematics teaching is the development of the mental discipline

- "**cultivating thinking power**" -

because "mathematics is the operation of elevating the potential of human reason with the power of reason"….

(Pestalozzi)

# The context

Students' responses suggested that the prerequisite mathematics courses fail to achieve some of our most cherished goals of education:

"Preparation for later life
Promoting cultural competence
Developing an understanding of the world
**Developing critical thinking**
**Developing a willingness to assume responsibility**
Practice in communication and cooperation
Enhancing students' self-esteem"*

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* H.W.. Heymann, 2003, *Why teach mathematics? A focus on general education*. Kluwer Academic Publishers.

# Question

If the prerequisite mathematics courses
do not nurture the sense of truth in students
and do not cultivate their thinking power,
why do we require that they be taken by candidates
to psychology, commerce, nursing, engineering
etc?

## Maybe there are some other goals of teaching mathematics?

# Goals of education from the point of view of the political economy of education

- Building human capital by teaching skills that directly enhance productivity
- Providing a screening mechanism that identifies ability
- Building social capital by instilling common norms of behavior
- Providing consumption good that is valued for its own sake

Gradstein, M. 2005: *The political economy of education*.

## Goals of education from the point of view of the political economy of education

Do we want to classify

"nurturing the sense for truth"

and

"cultivating thinking power"

as

a consumption good

that is valued for its own sake

?

# Mathematical knowledge for sale!

## 20% off

**BUY NOW!**

But let's not dismiss the political and economical perspectives on education with a joke…

Let's face the institutional reality of mathematics education

# Purposes of prerequisite mathematics courses from the point of view of university mathematics departments

- To equip students with basic algebraic and analytic techniques involved in mathematical models and methodological tools used in economics, psychology, medicine, engineering, etc.
- To provide a screening mechanism that identifies ability
- To provide a source of financial support for the mathematics departments who staff the courses with instructors, markers and tutors recruited from among the faculty, visiting professors and graduate students.

# Purposes of prerequisite mathematics courses from the point of view of university mathematics departments

But is it correct to tell students that they must take the prerequisite mathematics courses

because they will need the mathematics

in their target academic programs,

and then fail to develop

their **independence** as **critical** users of mathematical models?

# Purposes of prerequisite mathematics courses from the point of view of university mathematics departments

Do we have any use for financial advisors who are **not critical** with respect to the predictive power of the mathematical models they are using and blind to the mistakes they are making?

How credible are reports of psychologists who use statistical methods in their studies but **do not understand the theoretical assumptions** and limited applicability of the methods they are using?

How about engineers who **leave it to the users to check the validity** of their constructions?

# A question that triggered my recent research

Is it **possible**

to make the prerequisite mathematics courses serve **educational** as well as administrative and economical purposes by modifying the teaching approaches?

***Realistically*** possible - that is respecting the institutional constraints under which these courses function?

# Institutional constraints of the prerequisite mathematics courses

- The courses are short and intensive; e.g., the topic of inequalities with absolute value is allotted 20-40 minutes of class time in a pre-calculus course
- Classes are large
- Instructors are mathematicians or graduate students in mathematics, not trained as teachers
- Research is more important for them than teaching
- Instructors have rarely experienced anything another than lecture-based, chalk-and-talk courses at the university
- Assessment is mostly summative, not formative

# Reforms must respect institutional constraints

Theoretical and empirical studies of institutions warn against the belief in the existence of simple solutions of reforming them. They suggest that such systems develop characteristics that tend to perpetuate themselves (e.g. Crozier & Friedberg, 1980).

Any attempt to reform an institution or to explain why an attempt has failed must be based on understanding its functioning as a system.

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Crozier, M. & Friedberg, E. (1980). Actors and systems. The politics of collective action. (Chicago and London : The University of Chicago Press.)

# A hypothesis

If students in the prerequisite courses were lectured not only on rules, formulas and techniques of solving standard questions but also on some of the theoretical underpinnings of these,

then they would have more control over the validity of their solutions

and would be more interested in checking the correctness of their solutions.

# A teaching experiment

Three lectures on inequalities with absolute value

- PA – “procedural approach”
- TA – “theoretical approach”
- VA – “visual approach”

Substitution of concrete numbers used as a means of control (PA, TA)

Logical analysis used as a means of control (VA)

# Task-based interviews

Tasks:

- 1. Calculate: $||16 - 24| - |7 - 56||$
- 2. Solve: $|x - 1| < |x + 1|$
- 3. Solve: $|x + 3| < -3|x - 1|$
- 4. Solve: $|2x - 1| < 5$
- 5. Solve: $|2x - 1| > 5$
- 6. Solve: $|50x - 1| < |x + 100|$

# Task-based interviews

Interview questions, among others:

- How did you solve this question?
- How do you know your answer is correct?
- In general, when you do your assignments, or write examinations, are you checking your answers? If yes, how?

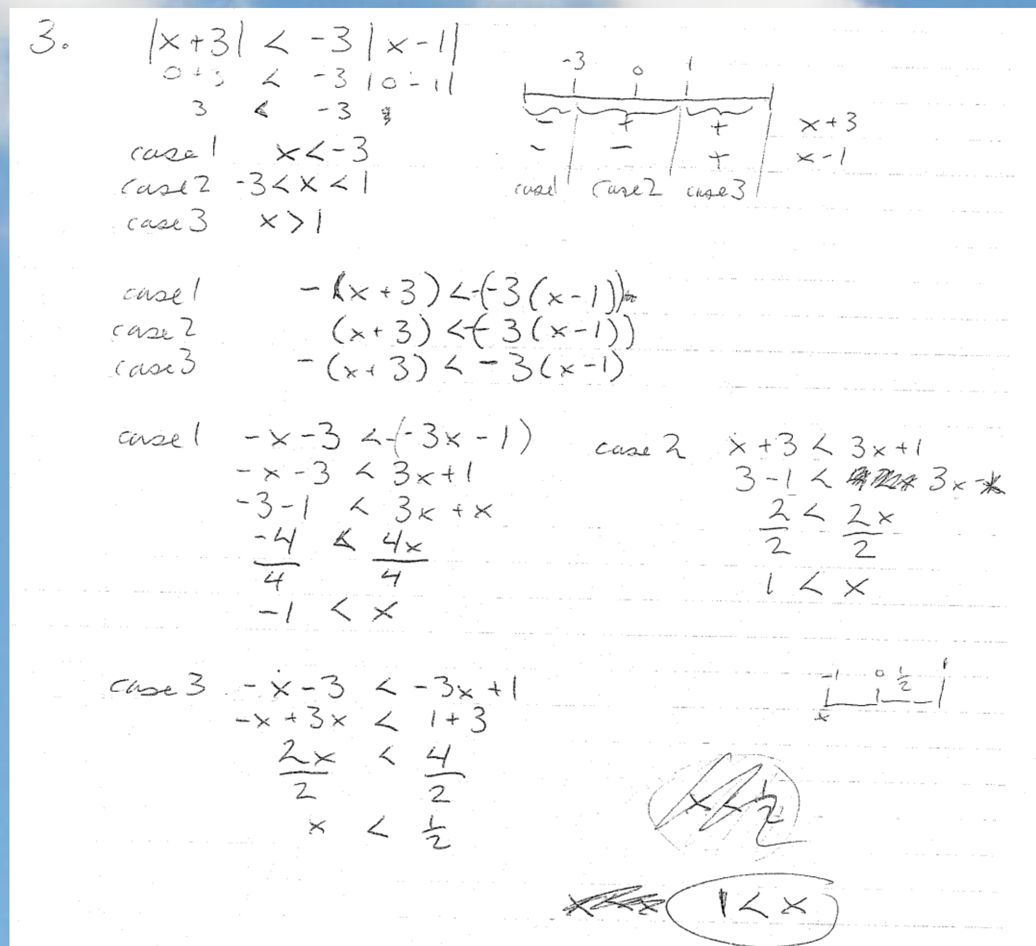
# Examples of students' solutions

- 18 students so far volunteered to participate in the teaching experiment
- 6 students watched the PA lecture
- 6 – the TA lecture, and
- 6 – the VA lecture

# Sarah

- After the PA lecture
- Sarah's method: imitation of the taught method
- No verification of results

# Sarah’s solution to Question 3



3. $|x+3| < -3|x-1|$

$0+3 < -3|0-1|$

$3 < -3$

case 1 $x < -3$

case 2 $-3 < x < 1$

case 3 $x > 1$



case 1 $-(x+3) < -(-3(x-1))$

case 2 $(x+3) < -(-3(x-1))$

case 3 $-(x+3) < -3(x-1)$

case 1

$-x-3 < -(-3x-1)$

$-x-3 < 3x+1$

$-3-1 < 3x+x$

$\frac{-4}{4} < \frac{4x}{4}$

$-1 < x$

case 2

$x+3 < 3x+1$

$3-1 < 3x-x$

$\frac{2}{2} < \frac{2x}{2}$

$1 < x$

case 3

$-x-3 < -3x+1$

$-x+3x < 1+3$

$\frac{2x}{2} < \frac{4}{2}$

$x < \frac{1}{2}$

$1 < x$

# Yufei

- After PA
- Used the method presented in the lecture
- Checked all her answers by numerical testing, mentally
- All her answers were correct.
- In Q. 3, she solved the inequality using the presented method, then noticed that she didn't have to do that, because there was a contradiction in the expression itself
- In Question 5 ($|2x-1|>5$) she said she suspected that the solution would be a complement of the solution to Question 4 ($|2x-1|<5$), but still solved it using the method presented in the lecture as a way of checking her conjecture.
- She was thus learning a theory while what she was doing appeared to be a mere "mechanical" application of the procedure.

# Phil

- After the TA lecture
- Phil's methods:
  - systematic numerical testing by choosing numbers in the neighborhood of solutions to equations derived from the inequality
  - qualitative visualization
- Difference between checking and solving blurred
- Very concerned about the truth of his statements
- Sensitive to contradictions in statements about concrete integer numbers, but had difficulty with fractions
- Noticed contradictions in algebraic statements (solved Q. 3 by reasoning)

# Phil – Question 4 – numerical testing in the neighborhood of solutions to equations

4. $|2x-1| < 5$

$2x-1=5 \Rightarrow 2x=6 \Rightarrow x=3$

$-(2x-1)=5 \Rightarrow -2x=4 \Rightarrow x=-2$

[illegible]

$|-2\cdot 2-1| = 5$

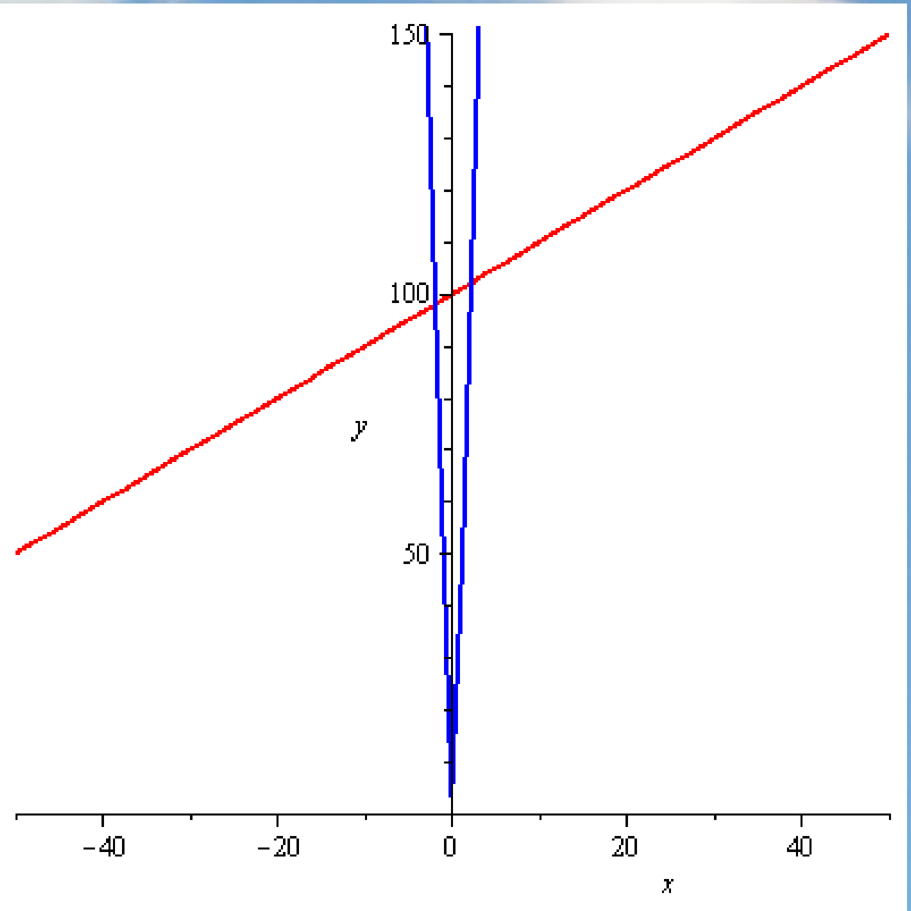
true for $x \in ]-2, 3[$

# Phil – question 6 – qualitative visualization |50x-1|<|x+100|

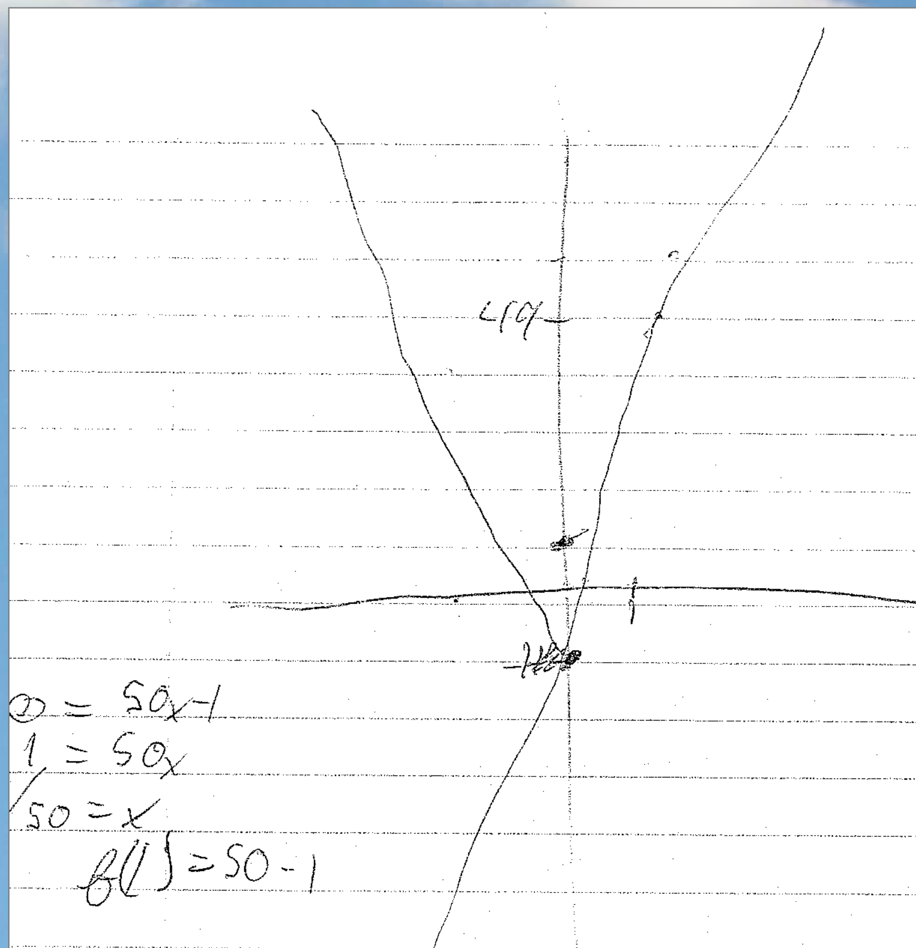
- Ph: Uh… So… here I tried a method similar to… uh… Question 2, trial and errors… uh… so if it’s 0, it doesn’t tell us much because -1, yeah, minus one is smaller than one hundred, but here, if we look quickly, 100 here, so how do we get 100 there? So, it’s using 2. Okay, so we might look 2, what… what’s happening, so 100 minus 1 it’s 99; and here, 2 plus 100, it’s 102. Okay. But if I use something bigger, because this here, if we write it as a function, it’s **evolving slowly**, because of 1, each time we have one to the x, but here in addition to the one to the x, we have 50, so we must be careful, because it **gets quickly bigger**…

Phil was asked to graph the functions in question 6: $|50x-1|<|x+100|$.

# We expected something like that:



# But this is what Phil drew:



# Phil's opinion about the separation between social sciences and mathematics in secondary school

Phil takes a Linear Algebra course because he thinks it will help him to understand economics better. He tells us this story:

" At the end of my high school I had bad experience with mathematics
and then I said, oh no, you know, math it's out of my life
and then I observed in *Political Science Review* – because I am more interested in political science – that, yes, you can have VERY interesting mathematical applications
and then if you completely avoid math, well, you got philosophy,
with it's pompous blah-blah, Plato revisited,
so okay, what kind of job you get in studying that?
Teaching? I'm sorry, but the answer is, No.
And if you make some policy analysis for efficiency and efficiency is a more economic concept and in economics you got more math.
But the problem is that, in high school, it won't really interest you, the math, because math teachers are more… uh… with the natural sciences teachers,
and so the applications of math in high school are not in economics because there's almost no courses in economics, it's physics and chemistry,
so in [upper secondary] school for [students interested in] social science, math is absolutely irrelevant.
Only later on, I observed that mathematics is a very important methodological tool."

# Frequency tables

# Results: The rate of correct answers

| | ***Percent of correct answers*** *among students following each lecture* | | | *Percent of correct answers in* ***total*** | *Lecture followed by* ***maximum*** *correct answers* | *Lecture followed by* ***minimum*** *correct answers* |
|---|---|---|---|---|---|---|
| ***Question*** | **PA (N=6)** | **TA (N=6)** | **VA (N=6)** | **All interviewees (N=18)** | | |
| 1. | 100% | 100% | 67% | 89% | PA, TA | VA |
| 2. | 50% | 50% | 83% | 61% | VA | PA, TA |
| 3. | 50% | 50% | 67% | 56% | VA | PA, TA |
| 4. | 50% | 67% | 83% | 67% | VA | PA |
| 5. | 67% | 33% | 83% | 61% | VA | TA |
| 6. | 33% | 17% | 33% | 28% | PA, VA | TA |
| ***Average correct*** | **58%** | **53%** | **69%** | **60%** | **5VA, 2PA, 1TA** | **4TA, 3PA, 1VA** |

# Results: The rate of correct answers

- The VA lecture, which presented two methods of solving inequalities with absolute value (the graphical method and the logical analysis method used also as means of control) was followed by the largest number of correct solutions.
- The TA lecture, which presented a very formal approach, strictly logical and algebraic, was followed by the smallest number of correct solutions.

# Results: Transfer

- Questions 3, 4 and 5 were different from the worked out example in the lectures. Therefore success on these questions could be a considered as a measure of the transfer of knowledge learned in the lecture.
- The average percentage of correct solutions to these questions among students who followed the PA lecture was (50%+50%+67%)/3 = 56%;
- among those who followed the TA lecture, it was (50%+67%+33%)/3=50%;
- among those who followed the VA lecture, it was (67%+83%+83%)/3=77%.
- **Therefore the VA lecture appeared to generate more transfer than the other two lectures; TA was the least promising in this respect.**

# Results: *Students' sensitivity to contradictions and concern with the validity of their solutions*

| | After lecture of type: | | | | | |
|---|---|---|---|---|---|---|
| | **PA (N=6)** | **TA (N=6)** | **VA (N=6)** | **Total (N=18)** | max | min |
| Percent of correct answers to Q. 3 | 50% | 50% | 67% | 56% | VA | PA, TA |
| Percent of students who checked their answers | 50% | 33% | 50% | 44% | PA, VA | TA |
| Percent of students who checked by: | | | | | | |
| Numerical testing | 33% | 50% | 33% | 39% | TA | PA, VA |
| Reasoning ("I proved it") | 33% | 0% | 50% | 28% | VA | TA |
| Graphing | 0% | 0% | 17% | 6% | VA | PA, TA |
| "I followed the taught method, so it's correct" | 33% | 17% | 0% | 17% | PA | VA |

Legend:

Question 3: $|x + 1| < -3|x - 1|$

# Results: *Students' sensitivity to contradictions and concern with the validity of their solutions*

- Responses to Question 3 might be an indicator of students' sensitivity to mathematical contradictions. The highest sensitivity was among VA students; PA and TA students were as likely as not to notice the contradiction. So PA and TA lectures may be said to have no impact on enhancing or hindering students' sensitivity to contradictions.
- TA students were rather unlikely to check their answers. But after PA and VA, they were as likely as not to check their answers, so we can say that none of the lectures was followed by students being more likely than not to check the validity of their answers.
- In fact, the difference between solving an inequality and checking the solution was quite blurred in the students. Numerical testing was used as a solution method by many students; they were testing and solving at the same time.
- Checking by reasoning was more frequent among VA students, but even here it was not very frequent (50%).

# Results: *Dependence on teachers*

- Predictably, thinking that “correct” means to just follow the method presented in the lecture, was more common among PA students.
- In fact, one can define “procedural learning” as having this notion of “correctness”: my solution is correct because “I did what I was supposed to do”
- But procedural learning in this sense was not very common among the subjects, which is an optimistic result.

# Results: *Relations between methods taught and methods used by students in solving the exercises*

| | *After lecture of type:* | | | | | |
|---|---|---|---|---|---|---|
| *Solving approaches used by students* | **PA (N=6)** | **TA (N=6)** | **VA (N=6)** | **Total (N=18)** | max | min |
| Numerical testing / random | 17% | 33% | 0% | 17% | TA | VA |
| Numerical testing / systematic | 17% | 17% | 17% | 17% | | |
| Numerical testing (any kind) | 33% | 50% | 17% | 33% | TA | |
| PA | 50% | 0% | 0% | 17% | PA | |
| TA | 33% | 67% | 50% | 50% | TA | PA |
| VA (TA+graphing) | 17% | 0% | 50% | 22% | VA | TA |
| Reasoning (not calculations) used in Question 3 | 17% | 33% | 50% | 33% | VA | PA |

Legend:
Question 3: $|x + 1| < -3|x - 1|$

# Results*: Relations between methods taught and methods used by students in solving the exercises*

- VA which uses logical analysis as means of control was followed by the most sophisticated methods.

- TA was mostly followed by the least sophisticated method (random numerical testing).

- The most frequently used method overall was TA (dull analysis of all logically possible cases) – 50% of students used it;
  - next ranking was numerical testing with 33%;
  - VA (logical analysis of cases mixed with the graphical method) was far behind TA with 22%.

- Some students were using TA as a procedure, not as a logical analysis. One student always followed his logical analysis with numerical testing because that's how it was done in the lecture. He did not always notice contradictions between the numerical testing and the results of the logical analysis solution.

## Results: *Relations between methods taught and methods used by students in solving the exercises*

- It is perhaps not true that the lecture approach had no influence on students' solutions, because the frequency of using the method lectured was higher than using any other method in each case.
- However, the frequencies were not high. One could say that, after PA or VA, students were as likely as not to use the method taught (50%).
- It is only after TA that students were more likely than not to use the method taught (67%).

## Results*: Relations between methods taught and methods used by students in solving the exercises*

- Using numerical testing instead of the method taught or as an additional method, was most frequent after the TA lecture.
- VA was followed by the highest frequency of using reasoning in Q. 3.
- PA was followed by the smallest frequency of reasoning in this question.

# Some conclusions from the frequency analysis

- VA was followed by
  - maximum correct answers
  - maximum 100% correct answers
  - maximum correct answers to Q. 3, so greatest sensitivity to mathematical contradictions
  - more transfer
  - more sophisticated methods of solving and checking one’s answers

# Some conclusions from the frequency analysis

- It may not matter very much which approach is used in the lecture for the strategies students then use in solving problems
  - So perhaps students depend on teachers for the validity of their solutions but, at the same time, they are resistant to teaching?
- None of the lectures was followed by students being more likely than not to check the validity of their answers.

# Questions

Why students

- depend on teachers for the validity of their solutions ?

- are not sensitive to mathematical contradictions ?

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Sierpinska, A.: 2007, “I need the teacher to tell me if I am right or wrong”, PME31, Seoul, Korea.
(can be downloaded from http://www.asjdomain.ca/

# Reflections on students’ dependence on teachers for the validity of their solutions

*There could be some good reasons why students’ depend on teachers for the validity of their solutions:*

- Epistemological reasons:
  - Much of mathematics is tacit knowledge
  - A mathematical concept is like a banyan tree
- Affective reasons
  - The school mathematics discourse uses expressions such as right / wrong – suggesting an authority decision - rather than true / false
- Didactic reasons
  - In didactic situations the task is given by the teacher
  - Many school tasks do not give the student the possibility to verify the answer on his or her own
- Institutional reasons
  - In school, validity = compliance with institutional rules and norms

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For details see my PME 31 talk, available from

http://www.asjdomain.ca/

# Reflections on students' lack of sensitivity to mathematical contradictions (LSC)

There could be equally good reasons why students are not very sensitive to mathematical contradictions:

- Epistemological
- Cognitive
- Affective
- Didactic
- Institutional  reasons

# Epistemological reasons of students' lack of sensitivity to mathematical contradictions

Contradiction depends on meaning

Contradiction presumes that there is meaning

Contradiction presumes there is rigor in definitions and reasoning

Concern about contradictions is more natural if one asks questions such as:

*Is this statement true? Is it consistent relative to the given conceptual system?*

rather than questions such as:

*Does this technique work? Is this answer acceptable in the given context?*

# Epistemological reasons of students’ lack of sensitivity to mathematical contradictions

*Contradiction depends on meaning*

Example:

$$-x < 2 \text{ and } x < -2$$

**Meaning 1:**

Conjunction of two conditions on the real variable x.

There is a contradiction.

**Meaning 2:**

Result of application of a theorem-in-action to a formal expression seen as a kind of diagram:

If $a < b$ and $c \neq 0$ then $a/c < b/c$

There is no contradiction

# Epistemological reasons of students' lack of sensitivity to mathematical contradictions

## *Contradiction requires meaning*

If a statement is meaningless for you,
the question of consistency does not exist for you

Absolute value is associated with situations where only the magnitude – and not the direction – of a change in a one-dimensional variable is evaluated. It may thus be seen as a particular norm, namely the two-norm in the one-dimensional real vector space R1.

$$|x| = \sqrt{x^2}$$

Absolute value is thus an abstraction from the sign of the number and can be defined also as:

$$|x| = \begin{cases} x & if \quad x \geq 0 \\ -x & if \quad x < 0 \end{cases}$$

This notion is meaningless if the student's idea of number is magnitude. There is nothing to abstract from.

# Affective reasons of LSC

- Relying on “gut feeling” about one’s solutions: I know it’s correct “if everything goes smoothly”, “solution is neat”, “numbers are nice”
- Not verifying one’s answers for fear of losing morale, self-confidence, on a test

# Didactic reasons of LSC

Depriving students of opportunities for noticing a contradiction for the sake of “fairness” of assessment

# Institutional reasons of LSC

*In school, validity = compliance with institutional rules and norms (didactic contract)*

In school practice, mathematics becomes, in fact, a collection (often a loose collection) of types of tasks (exercises, test questions, etc.) with their respective techniques of solution, where the form of presentation (e.g., “in two columns”) often has the same status as the mathematical validity. Techniques are justified on the basis of their acceptability by the school authorities, not on their grounding in an explicit mathematical theory. It is not truth that matters but respect of the rules and norms of the didactic contract related to solving types of problems.

# A final remark

In mathematics education we commonly blame students' poor knowledge of mathematics and negative attitudes to its study on procedural approaches to mathematics teaching and we claim that mathematics taught that way is not worth teaching or learning.

# A final remark

We constantly call for reforms that would support conceptual approaches to mathematics teaching.

I, too, blamed students' dependence on teachers for the validity of their solutions and their lack of sensitivity to contradiction on the "rote model".

# A final remark

But what guarantee is there that those ills would be removed by adopting a particular approach to teaching mathematics?

A lot of money and human effort could be spent on implementing the desired model and the results might be quite disappointing.

The expected students' interest, autonomy and mathematical competence might not materialize not because of lack of teachers' competence or good will but because of epistemological, cognitive, affective, didactic and institutional reasons that are independent of their knowledge and good will.

# A final remark

These reasons have their roots

in the nature of mathematics,

in human nature,

in the very definition of a didactic situation,

and in what makes a school a school rather than a Montessori kindergarten.