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PERSPECTIVES ON RESEARCH IN MATHEMATICS EDUCATION[1]

ABSTRACT. The paper is a review of chosen approaches to research in mathematics education in several countries: Germany, France, United Kingdom, United States, Russia, Poland, and Canada. The review is done in the literary form of a satire, in which a character is taken on a voyage to a variety of "islands" representing different research interests and methodologies in mathematics education. The story is a parody of Homer's *Odyssey*, and the main character is called Odysseus. Odysseus' role is played by the famous arithmetic problem about a team of an unknown number of reapers who are given the task of mowing down two meadows one of which is double the size of the other. As the problem travels from one "island" to another, mathematics educators do different things to and with the problem and it is solved is a variety of ways. The main text of the paper reads as a story and there are no explicit references and names of authors, whose work is only alluded to. However, the solution to all allusions, i.e. explicit references, can be found in the footnotes.

> ÒWhatÕs the use in having mathematics all the time, and writing? Better tell us something, about the earth, or even history, and we will listen,Ó say all.
> L. N. Tolstoy, *The School at Y snaya Poly na*. [2]

This talk is about certain approaches to research on the teaching and learning of mathematics that can be found in some European and North American countries. It was first written (in French) at the occasion of the celebrations related to World Mathematical Year 2000. To underscore the festive mood of the event, the form of a satire rather than that of an academic discourse was used. It is a story whose main character is the famous problem about a team of an unknown number of reapers who are given the task of mowing down two meadows one of which is double the size of the other. The story parodies Homer's Odyssey; the problem plays the role of Odysseus who goes on a voyage and visits a variety of "islands" representing different research interests and methodologies in mathematics education. As the problem travels from one "island" to another, mathematics educators do different things to and with the problem and it is solved is a variety of ways. The main text of the paper reads as a story and there are no explicit references and names of authors. These can be found in the footnotes.

There is a legend saying that the "reapers problem" was a favorite of Lev Nikola vitch Tolstoy (author of *Anna Karenina* and *War and Peace*) who liked to give it to children and illiterate peasants in his school at Y snaya Poly na [3].

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[1] This paper is a third version of a text, which first appeared as, Sierpinska, A. (2000): Mathematics ducation et Didactique des math matiques - Quelle dif f rence˚? - Actes du 42e congr s annuel de lÕAssociation Math matique du Qu bec, Sainte-Foy, Qu bec, 19-31 octobre 1999, p. 131-154. The second version modified and extended the first; it appeared in Zentralblatt f r Didaktik der Mathematik 34 (4), 164-174. The present text is largely an English translation of the second version, but contains several important modifications.

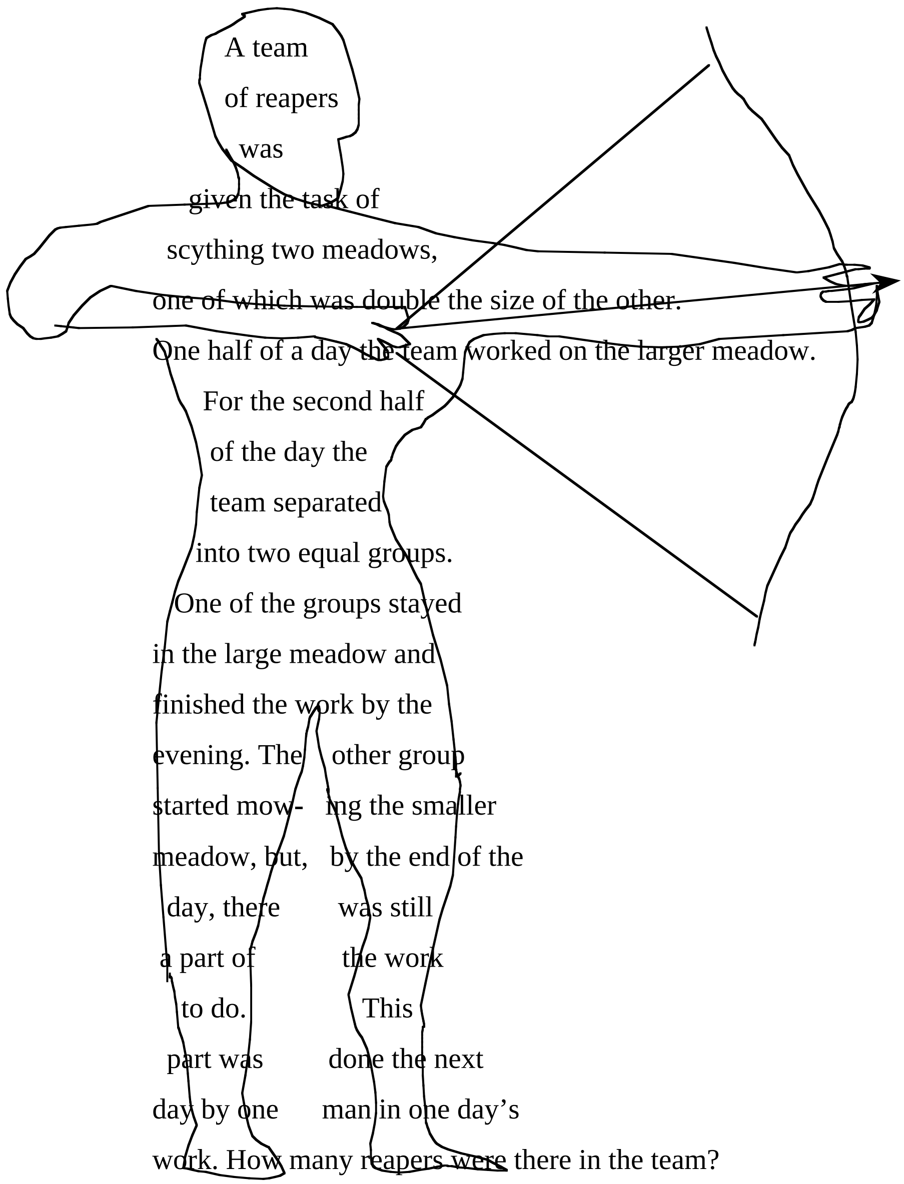
[2] Tolstoy, L.N. (1967): Tolstoy on Education. Translated from the Russian by Leo Wiener. - Chicago and London: The University of Chicago Press, p. 227-360

[3] Perelman, Ya. I. (1979): Algebra can be fun. - Moscow: Mir

ΔΥΔΑΚΤΕΙΑ

OR THE WONDERFUL JOURNEY OF THE ASTUTE PROBLEM OF REAPERS

Let me first introduce to you Odysseus, *man of many wiles*[4]:



A team of reapers was given the task of scything two meadows, one of which was double the size of the other. One half of a day the team worked on the larger meadow. For the second half of the day the team separated into two equal groups. One of the groups stayed in the large meadow and finished the work by the evening. The other group started mow- ing the smaller meadow, but, by the end of the day, there was still a part of the work to do. This part was done the next day by one man in one day's work. How many reapers were there in the team?

Imagine our Odysseus' Troy to be the "New Math war", which had shaken Europe and America in the 1960s. In this war Algebra was put against Arithmetic and Geometry. Algebra didn't exactly win the

[4] Odysseus is characterized as "man of many wiles" in the Odyssey's translation byAllen Mandelbaum (1990; University of California Press, Berkeley). All citations from Odyssey used in the present paper come from this edition. They are marked by the use of italics.

war; but neither did Arithmetic or Geometry who lost their innocence in the fight. Odysseus is a survivor of the war and one of its heroes: he challenges Algebra and gives new meaning to Arithmetic.

Let Algebra be our Odysseus' Poseidon ("Algebraidon" will be this god's name), who will relentlessly chase him across the islands of the "Didactic Archipelago" before he can finally find the way to *his own dear land.* This will be the story of Odysseus' escapes from Algebraidon's fury.

1. Odysseus at the epistemological triangle

As Dawn's rose fingers touched the sky, Odysseus' awoke on his fine ship, beached along the white sands of the "Epistemological triangle Archipelago". No sooner did he set foot on the land that he was surrounded by the messengers of Algebraidon and dragged, against his will, to a mathematics classroom. Algebraidon, disguised as the teacher, threw him to a gang of adolescents. In the last week, the class worked on solving linear equations.

This was no ordinary classroom. The teacher was a student teacher, doing her professional stage in the school. There were microphones everywhere, a cameraman was filming the lesson, and, from the back of the classroom, three serious gentlemen were observing the classroom interactions and taking field notes. But they observed different things and thought different thoughts.

The first gentleman was in a *philosopher*'s mood. He was asking himself: "What am I doing here? The report I'll write after this lesson, will it be an account of a reality or of my perception of the reality? If, later, I try to explain my observations by constructing a theory, what will be the nature of this theory? Will it be a scientific or, in other words, a falsifiable theory? Perhaps not. Perhaps this theory will rather belong to hermeneutics or the art of interpretation of texts since, after all, all I'll have after the lesson will be a protocol, which is a text."[5]

The comportment of the second gentleman was much more lively. He would give a jump in his chair each time he recognized a familiar pattern of interaction between the teacher and the children. Look, there he goes again, noticing that the teacher is using the well-known "funnel" interaction pattern in leading children to formulate an expected solution.[6].

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[5] Questions about the epistemological and institutional status of research in mathematics education have interested several researchers, one of whom was Hans-Georg Steiner, evoked here in the character of the Philosopher. Concerning these questions, see, for example, the collection of articles in:
Sierpinska, A. & Kilpatrick, J. (Eds.) (1998): Mathematics Education as a Research Domain: A Search for Identity. An ICMI Study. - Dortrecht: Kluwer Academic Publishers

[6] Bauersfeld, H. (1978): Kommunikationsmuster im Mathematikunterricht Ñ Eine Analyse am Beispiel der Handlungsverengung durch Antworterwartung. — In: H. Bauersfeld et al. (Eds.), Fallstudien und Analysen zum Mathematikunterricht. Hannover: Schr del, p. 158-170
Voigt, J. (1995): Thematic Patterns of Interaction and Sociomathematical Norms. - In P. Cobb and H. Bauersfeld (Eds.), The Emergence of Mathematical Meaning: Interaction in Classroom Cultures. Hillsdale, N.J.: Lawrence Erlbaum Associates, p. 163-202
Wood, T. (1998): Alternative Patterns of Communication in Mathematics Classes: Funneling of Focusing? — In: H. Steinbring, M.-G. Bartolini-Bussi, A. Sierpinska (Eds.), Language and Communication in the Mathematics Classroom. Reston, VA: NCTM, Inc., p. 167-178

Teacher: So, children, how are we going to solve this problem? Lisa?

Lisa: First, I assumed that there were 4 reapers, so, in the morning they mowed together 4 patches like this of the larger meadow. (*Lisa draws four rectangles on the board*). Then, in the afternoon, there was only half of the team in the meadow, that is, two reapers. So they mowed two more patches like this. (*Adds two more rectangles*).

Teacher (*interrupts*): But how can you presume there were 4 reapers? That's exactly what we don't know! And when we don't know, what do we do, children? (Silence) When we don't know(*stressed*), this means that the value is (*suspends her voice in expectation of an answer*)?

Kai: an unknown!

Teacher: Very good, Kai! We put down an unknown. What letter shall we use to name it?

Several students: x!

Teacher: Very good: x. So x will be the number of reapers. There is one more thing that we don't know in this problem. It's (*suspends her voice and looks around at the students in expectation*).

Students: If the reapers were all as good at mowing, and there weren't any lazy ones among themÉ

Teacher (*looks unsatisfied*): MmmÉ

Kai: How big the larger meadow was, the number of square metersÉ

Teacher (*looking contented*): Exactly! The area that each reaper was scything in half a day. Let's put a for this quantity. You shall eventually see that this variable is not very important, but it will be useful in writing the equation. Okay, so what is the equation? In the morning, x men have mowed each an area of a. How much have they mowed altogether?

Several students: a times x!

Teacher: Good! (*She writes* ax *on the board*). In the afternoon, there was only half the teach, so (*voice raised and suspended in expectation*)

Kai: One half of x times a!

Teacher: (*adds* + 1/2 ax *and gets* ax + 1/2 ax) In the smaller meadow, one half of the x men mowed each an area of a during the rest of the day (*puts 1/2ax on the same line but further to the right*). And the next day, one reaper mowed the remaining part in one day. So, if one man mows down an area of a in one half of a day, what is the area he mows in one day?

Kai: 2a.

Teacher: (*puts +2a to the right of the previously written expression*). But we are told that the larger meadow is twice as big as the smaller one, so what equation will we obtain?

Togba: Two times the left side equals the right side.

Teacher: (*harshly*) You always make the same mistake, Togba!

Kai: We have to put the 2 on the right side.

Teacher: Of course! (*She completes the equation which becomes* ax + 1/2ax = 2˚(1/2ax + 2a)*; she then asks Togba to come to the board and solve the equation).*

The "interactionist" gentleman obviously enjoys the episode; he can add it to his collection of examples of instruction, which, without teaching the students anything, manages nevertheless to pull the right answers out of them!

The focus of the third observer is *epistemology*. He is as lost in thought as the first, but he seems a little bit sad. He recalls what his Master told him one day about the problem of reapers. For him, it was not necessary to use equations to model the situation but it was not enough to think about numbers only as an instrument for counting things. This problem highlighted the relational character of numbers in the way it used fractions to describe the relations between quantities without giving any other information about these quantities.[7] The sad thing about the lesson was, however, that the teacher

[7] Otte, M. (1981): What Relevance has the ÒProblem of TextsÓ for Mathematics Education and its Understanding? Occasional Paper 15. - Bielefeld: Universit t Bielefeld / IDM

evacuated this epistemological value completely from the problem. In a way, the relational concept of number has been replaced, in the lesson, by an algebraic form and reduced to a technical manipulation of symbols, without any link with the object represented by this algebraic form. Without this link there could be no emergence of the concept of number, because a concept is always a relation between a symbol and the object it refers to or a context where it can be applied.[8].

After the lesson, Odysseus effortlessly quit the students' memory and managed to leave the classroom almost unnoticed, taking advantage of a moment of confusion when Algebraidon, under the appearance of the teacher, found himself attacked by the Epistemologist.

2. Odysseus in the land of games and Paradoxes

When Dawn's rose fingers reached the sky, Odysseus arrived into a beautiful city surrounded by vineyards. He sent *two of his crewmen and a third, who served as a herald, to see what sort of mortals held this land. Those three were quick* to find out the following.

This land was home to a people of game-players. The stake in the games they constantly played was what they called "Le Savoir Mathematique". They distinguished it sharply from "Connaissances Mathematiques", which was something they personally invested by engaging with these games. It was not clear for Odysseus' men to see the difference between the two, so they asked for an explanation, but the game-players couldn't agree on one and entered into an interminable argument among themselves. They soon *disremembered* that some strangers were waiting for their response. So Odysseus' men moved on and noticed some kind of commotion in one corner of the land. A group of people appeared to be staging a piece of theatre; they were talking about "staging a situation", but it could be a *lapsus* because they often used the words "actor" and "paradox". So Odysseus' men thought that a comedy of errors was about to be played soon. Upon learning about the arrival of Odysseus, the group proposed to "set him up in a situation" ("de le mettre en situation", in the local dialect). All crewmembers were invited to participate in the show.

And so they all went to a large but not very high concrete building, called *The School* or *The Institutionalized Situation of Studious Leisure.*[9] The place was resounding with the voices of several hundred children, engaged in a variety of roles such as The Universal Subject, The Generic Student,

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[8] Steinbring, H. (1999): Reconstructing the Mathematical in Social Discourse - Aspects of an Epistemology-based Interaction Research. — In: O. Zaslavsky (Ed.), Proceedings of the 23rd Conference of the International Group for the Psychology of Mathematics Education, Vol. I. Haifa, July 25-30, 1999, p. 40-74

[9] Bourdieu, P.: 1994, Raisons pratiques, Sur la th orie de l'action. — Paris: Editions du Seuil, p. 216. Bourdieu, an important reference for French didacticians of mathematics, derives the meaning of the word "school" from the Greek "schole" which can mean "leisure". He describes school as "*situation institutionalis e du loisir studieux*". School knowledge, indeed, is founded on the "scholastic view" (expression of Austin), which suspends all questions of existence and all practical intentions. It is a "socially instituted situation in which one can defy or ignore the common alternative between playing and being serious, by playing seriously or taking seriously ludic matters, and considering problems that serious people and really busy people passively or actively ignore" (ibid., my translation from French).

The Learning Subject, The Acting Subject, The Object Subject.[10] They played all kinds of games among themselves or with adults called Teachers or yet with the Milieu, organized for them by the Teachers.

It was rather difficult for Odysseus to make sense of the rules of those games, because nobody wanted to talk about them, probably because these rules would change as soon as they were made explicit. A whispered message caught the ear of Odysseus, spread, no doubt, by some reactionary elements, that there is, indeed, a contract governing each game, but it is important to break it because otherwise nobody will ever learn anything in that School. Odysseus noticed that there were two kinds of Teachers. Some Teachers were nice to children; they talked to them, applying more or less *ostensible* practices[11]. Other Teachers turned away from the children, pretending not to see them. Someone explained that the former teachers were working under the contract of *Didactic Situations* and the latter — under the contract of *A-Didactic Situations.*[12]

Finally, Odysseus and his companions were led to a kind of theater where the stage was separated from the audience by a glass window that made the audience invisible for the actors. Odysseus was taken to the stage while his men stayed behind the glass window. Odysseus was immediately surrounded by some twenty children who jumped on him like wolves ravenous for Mathematical Knowledge. Small packs of them attacked him from all sides, trying to find an optimal solution strategy and win 4 points.

Odysseus was a complete stranger for the Teacher and this put him ill at ease. He was supposed to act an A-didactic Situation and *pretend* not to know how to solve the problem. But he acted it badly because he *really* didn't know how to solve it. "Actor's Paradox be damned!" — he swore under his breath.

Children in one group were cutting Odysseus into little rectangles and discussing.

Subject-Actor[13] 1: Suppose there were 4 reapersÉThey worked half a dayÉ
Subject-Actor 2: And each of them mowed a patch like this of the meadow (*draws a little rectangle*)
Subject-Actor 3: So they mowed, together, four rectangles like this (*draws three more rectangles in the same line*)
Subject-Actor 1: In the afternoon, they separated into two equal groups, so there were 2 reapers in each group, two in the larger meadow and two in the smaller one.
Subject-Actor 2: (*adds four more rectangles to the picture, two in the same line as the first four, and two below*)
Subject-Actor 3: That's what they did in one day. But there is still a part of the small meadow to do, because it is one half of the large one, and here, it's only one third.
Subject-Actor 2: This part was done by one man in one whole day, so we have to add two more rectangles.
Subject-Actor 3: Yeah, but then this wouldn't be a half of the large meadow! This doesn't work!

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[10] Brousseau, G. (1997): Theory of Didactical Situations in Mathematics. — Dortrecht: Kluwer Academic Publishers (see pages 248, 280)
[11] Salin, H.: 'Les pratiques ostensives dans l'enseignement des math matiques comme objet d'analyse du travail du professeur'. — In: P. Venturini, C. Amade-Escot and A. Terrisse (eds.), Etude des Pratiques Effectives: L'approche des Didactiques. Grenoble: La Pens e Sauvage ditions, pp. 71-81
[12] Brousseau (1997)
[13] Brousseau (1997), p. 248

Subject-Actor 1: It's because we started with four men. It can't be four men. Five, perhaps?
Subject-Actor 2: Impossible! How can you separate five men into two equal groups!
Subject-Actor 1: Six?
Subject-Actor 3: Six doesn't work, either, because then the large meadow is made of nine rectangles and so the small meadow would have to be made of four and a half rectangles. But the rectangles are not divisible; they are units.
Subject-Actor 1: So it's eight maybe? (*Children check it; it works*)
Children (*turning into Generic Students*): Sir! We found it, we solved it!

Behind the glass window, emotions were rising. A discussion was started.

Spectator 1: This reminds me of the epistemological obstacle[14] we were talking about at the National Seminar. The way these children solved the problem resembles the method of false position used by the Egyptians, as we saw it in the Rhind Papyrus. This technique was quite useful to solve a certain kind of arithmetic problems, but its development stopped there, because it was very difficult to formulate a theory of this technique; the possibility of its application was dependent on the context of each problem. For the problem that was given here, the children started off as if using the method of false position, but they did not continue with the proportional reasoning, as one would expect; they re-started with a new value. So they ended up using a trial and error strategy, rather than the method of false position. Too bad. But, anyway, we have an example of an obstacle here: a way of knowing that works well with some problems but which is conceived of as a universal method and becomes a habit of thought. If these children started to believe that all problems of arithmetic could be solved this way, this would become an obstacle. One could perhaps see the symptoms of this obstacle by changing the problem a bit. If you let the smaller meadow to be 2/5 of the larger one and assume 12 reapers were needed to finish the job the next dayÉThat would give, that would give, let me see, 240 men in the team. Children could persist in doing the same thing, trying out all even numbers. But to get to 240 by trial and error, one would have to be very patient, indeed! This could stimulate some children to overcome the obstacle and try a new method.

Spectator 2: From 8 to 240, what an *informational leap*, indeed!![15] You are talking about changing the variables of the situation, so that children are forced to look for another strategy and thus put other ways of knowing into play. But what are the ways of knowing that are specific for this situation? A priori the original problem appears ideal for forcing students to use the relational concept of number. The most natural way of thinking about the problem seems to be in terms of fractions and fractions are, in fact, an expression of relations between magnitudes. That's how one could reason: the area mowed in the afternoon is 1/2 of the area mowed in morning, hence 1/3 of the whole area of the large meadow. The area of the smaller meadow remaining to mow the next day is thus 1/2 — 1/3 = 1/6 of the large meadow. Since this part was done by one man in one day, 1/12 was done in half a day. So in half a day, one man mows 1/12 of the large meadow. By dividing the 2/3 of the large meadow that was mowed in half a day by 1/12, one obtains 8, so there were 8 reapers in the team. But I was disappointed in observing the class, because children were able to solve the problem without using fractions. Fractions, never mind the relational concept of number, are not epistemologically necessary to solve the problem. I am wondering now if it is realistic to hope that one could ever get children to use fractions by changing the variables of the situation, as you propose.

The two spectators became absorbed into a new *a priori analysis*[16] of their situation.

Odysseus' companions became emotional for a very different reason. Horrified by seeing their captain ripped apart into little rectangles, they rushed to his rescue. Taking advantage of the recess, the patched him together and took flight from the School. Too much Studious Leisure at a time may not be good for your health!

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[14] Brousseau, G. (1983): Les obstacles pist mologiques et les probl mes en math matiques. — In: Recherches en Didactique des Math matiques 4 (No. 2), p. 165-198
[15] ibid.
[16] Artigue, M. (1989): Ing nierie didactique. — In: Recherches en Didactique des Math matiques 9 (No. 3), p. 281-308
Artigue, M. & Perrin-Glorian M.-J. (1991): Didactic Engineering, Research and Development Tool: Some Theoretical Problems linked to this Duality. - In: For the Learning of Mathematics 11 (No. 1), p. 13-18

3. Odysseus at the Belvedere

As soon as Dawn's rose fingers reached the sky, Odysseus arrived on a vast island. In the middle of the island there was a magnificent mountain, reverently called, by the natives, The Mount of Mathematics or *The Belvedere*.[17] Contrary to the people of The Epistemological Triangle, the inhabitants of this land believed in only one God — Mathematics — and in the possibility of having a unique coherent religion that could explain all phenomena of teaching mathematics. On the pinnacle the local Wise Man was seated. What made him wise was the unconstrained view, from his position, of the whole insular reality of the Didactic Archipelago.

Noticing Odysseus' uncertain ascent, the Wise Man asked him: *Who are you? From what family? What city?* Odysseus responded in these words:

Odysseus: I am an arithmetic problem. I am traveling the world chased by the fury of God Algebraidon who wants to reduce me to a mechanical calculation on algebraic symbols and thus deprive me of all meaning. I want to be reasoned; I belong to the oral culture, not to the written culture. The peasants of Yasnaya Polyana that I used to entertain in my youth would not touch me with a goose quill. They counted on their fingers or with pebbles explaining what each gesture represented and they never lost control over the operations they were performing.

The Wise Man: Why do you put the oral against the written and why do you think this opposition is parallel to the one between reasoning and mechanical calculation? Nobody would deny that mathematics is based on reasoning and yet it is the existence of a notation very different from speech that makes mathematical thought and mathematical operations possible[18]. Mathematics is not *oral speech that has been written down.*[19] I know that one can solve you orally, but there is a long, long way between doing that and doing mathematics!

Odysseus: So you are saying that if one solves me *modo arithmetico*, in one's head, one hasn't done mathematics yet?

The Wise Man: Probably not, for at least two reasons. Firstly, while it is true that, historically, *the first activities of counting employed an ample variety of material objects, such as drawings or gestures, and the first instances of deductive reasoning in geometry were realized on graphical objects traced in the*

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[17] Chevallard, Y. (1991): Postface: Didactique, anthropologie, math matiques. - In: Y. Chevallard et M.-A. Johsua, La transposition didactique du savoir savant au savoir enseign , avec un exemple dÕanalyse de la transposition didactique. Grenoble: La Pens e Sauvage ditions, p. 199-233 (see p. 233)
See also:
Sierpinska, A. (1995): Some Reflections on the Phenomenon of French didactique. - In: Journal f r Mathematik-Didaktik 16 (No. 3/4), p. 163-192
[18] Goody, J. 1977: La raison graphique. La domestication de la pens e sauvage. - Paris: ditions de Minuit, p. 213; c ited by M. Bosch et Y. Chevallard, in:
Bosch, M. & Chevallard, Y. (1999): La sensibilit de lÕactivit math matique aux ostensifs. Objet dÕtude et probl matique. - In: Recherches en Didactique des Math matiques 19 (No. 1), p. 77-123 (voir p. 101)
[19] Bosch et Chevallard, ibid, p. 100

sandĘ one should not forget that, beginning at least with Vi te, mathematics progresses by way of written symbolism, so that one can follow almost all the history of this progress without leaving the register of written language.[20] But School always believes, like you, that the manipulation of written inscriptions in mathematics is reduced to a *mechanical* activity. It is believed that the student proves he "knows what he is doing" only when he produces figures, diagrams and accompanies all that with an oral discourse or an oral discourse that has been written down. The second reason is that to solve an isolated problem, like you, is to solve a puzzle, not to do mathematics. If you want to become a mathematical problem you have to become a part of a whole mathematical *praxeology*.[21].

Odysseus: AÉwhat? Of a mathematical structure, you mean? I have always considered myself "astute", but I see I am astute enough to understand what you mean.

The Wise Man: The idea of structures is considered a bit antiquated these days. We have a much larger view today. We don't see ourselves as mathematicians but as anthropologists of mathematics. Our object of study is still mathematics and we still believe that the problem of mathematics teaching must be posed not in terms of the cognitive activity of the learner (this belongs to psychology) and not in terms of the actions of the teacher (which belongs to pedagogy), nor, for that matter, in terms of the social interactions among the students and the teacher (which belongs to sociology), but in terms of *the mathematical knowledge that the teacher and the students are supposed to be studying together*. But the anthropological perspective on mathematical knowledge allows us to see it in a much larger frame of mathematical practices in all kinds of social institutions.[22] From this point of view, each object of knowledge is defined as an element of a praxeology, understood as a system whose basic elements are (a) a set of tasks recognized as important for the institution, (b) the techniques and know-how needed to accomplish these tasks, (c) a technology, i.e. a description and justification of these techniques and know-how, and (d) a theory justifying, on its turn, the technology. If you want to become part of mathematical knowledge, you need to define yourself as an element of a mathematical praxeology.

Odysseus: I really don't know how to go about it. All this theory intimidates me a little. *Batiushka* Tolstoy did not pretend he was creating a theory of instruction; in fact, his point of view was fundamentally anti-theoretical. He only had a general philosophy of life and his experience in teaching peasants' children in *Y snaya Poly na.* [23] He strongly criticized the theory and practice of education both traditional and modern. He insisted on characterizing education as a process of the liberation of

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[20] Bosch et Chevallard, ibid., p. 103

[21] Chevallard, Y. (1997): Famili re et probl matique, la figure du profeseur . - In: Recherches en Didactique des Math matiques 17 (No. 3), p. 17-54

[22] ibid, p. 79

[23] Archambault, R.D. (1967): Introduction. — In:Tolstoy on Education. Translated from the Russian by Leo Wiener. Chicago and London: The University of Chicago Press, p. v-xviii (voir p. vi)

the individual who would be led to creative improvisation through understanding.[24] The aim of education, for him, was not socialization and preparation of the young for jobs and managerial positions in the society. The aim of education was to maintain and develop a culture, a civilized society.[25] Of course, one might turn all that into a theory. One could say, for example, that the educational activity of Tolstoy was part of an institution, namely of the informal institution of the movement, quite widespread at the time, of "carrying the lights of education" to the illiterate peasants. The instructors, largely recruited from among land owners, organized community centers or "people’s schools", where people could gather and learn to read and to write, to count and reason. Writing was often an inaccessible skill for the adults; it was therefore necessary, for the teaching of arithmetic and reasoning, to invent problems that did not require any writing. I was one of these problems. One could perhaps say that I belong to the domain of pedagogy. Do I belong to the domain of mathematics? Certainly not to the domain of academic mathematics; university is not interested in solving problems like me.

The Wise Man: I think you have a serious identity problem there, Odysseus, and you need professional assistance. Since you seem to be familiar with the language of structures, I think I can help you with a technique of analysis developed by a colleague of mine from the island of Conceptual Fields. We don’t agree on all points, but we both have a preference for models of mathematical knowing that award an essential role to mathematical concepts themselves.[26] Using his approach, I would say that you are a multiplicative structure of the type "product of measures".[27] Three measure spaces are at play here: M_1 = [reapers], M_2 = [work days], M_3 = [reaped areas]. Reapers can be counted in fractions of the whole team, represented by number 1. Also reaped areas can be measured in fractions of the area of the large meadow. The relations between these three spaces can be modeled by a bilinear function:

$$f : M_1 \times M_2 \to M_3$$

defined by:

f(*x* reapers, *y* work days) = area mowed by x reapers in y work days

Let n be the number of reapers in the team.

The assumptions of your problem yield:

$$f(1, 1/2) + f(1/2, 1/2) = 1$$

$$f(1/2,1/2) + f(1/n, 1) = 1/2$$

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[24] ibid., p. ix

[25] ibid.

[26] Vergnaud, G. (1990): Le r le de lÕenseignant la lumi re des concepts de sch me et de champ conceptuel. — In: M. Artigue et al. (Eds.), Vingt ans de didactique des math matiques en France. Grenoble: La Pens e Sauvage ditions, p. 177-191 (voir p. 146, cit dans Bosch et Chevallard, ibid, p. 117)

[27] Vergnaud, G. (1983): Multiplicative Structures. — In: R. Lesh and M. Landau (Eds.), Acquisition of Mathematics Concepts and Processes. New York: Academic Press, p. 128-175.

Bilinearity of f applied to the first equation implies f(1,1)˚=˚4/3, which, substituted to the second equation yields n˚=˚8. *Et voil* !

Odysseus: Misery, Algebraidon strikes again! I have to run, but let me ask you one more question: At the Epistemological Triangle the Epistemologist was talking about Object and you have used that same word, but did you mean the same thing? I feel all confused and dizzy from all these words, words, and wordsÉ

The Wise Man: Oh no, we don’t refer to the same idea. For him, Object is the attribute of a sign; it is a semiotic object, the reference, the context of using a sign. My Object is a unit of knowledge; object is anything that is recognized by an institution as an object. In fact, anything can be an Object!

These were his words. *Odysseus’* spirit *was* nearly *broken*, but he found the strength to get back to his ship and sail away. He decided to seek respite from Algebraidon’s attacks North of the shores of the *sweet*[28] land of Cartesian planes.

4. ODYSSEUS IN THE LAND OF CLUBS

When Dawn’s rose fingers reached the sky, a resplendent view of white cliffs lying towards the sea appeared to Odysseus’s eyes. The ship was *moored within a river mouth* and Odysseus sent *two of his crewmen and a third, who served as a herald, to see what sort of mortals held this land*. Upon his return, the herald said: "People in this land have no respect for a single king; everybody wants to be the lord and master on their own domain, no matter how small. The only thing that seems to unite them is their aversion to the monster called ’The National Curriculum’, which, apparently, wants to impose upon them a single correct vision of mathematics and its teaching. They are not used to it. Their society is organized into what they call ’clubs’ which are classes of abstraction of the relation ’same cup of tea’. I’ve seen people passing by an entrance to a club, mutter scornfully, ’it’s not my cup of tea’, and go on to another club. There is a large variety of these clubs or ’cups of tea’, if you will. Would you like to go and visit some of them, Odysseus?" Odysseus agreed and they took the road, making sure of wearing socks with their sandals, according to the local tradition.

The sign on the first club they visited showed something resembling a helix, whose coils wore inscriptions such as "Getting started", "Getting involved", "Mulling", "Keeping going", "Insight", "Being skeptical", "Contemplating".[29] Odysseus and his companions were ushered into the club by an elderly white-bearded character.[30] His name was Calebus[31]. Using discrete hints, Calebus drew Odysseus’ and his companions’ attention to the maxims painted on the walls of the club. These maxims encouraged the students to *shift their attention* between the different aspects of a problem, in order to

[28] La "douce France"
[29] Mason, J. with L. Burton and K. Stacey (1982): Thinking Mathematically. —Addison-Wesley, London (see p. 136)
[30] Allusion to Graham Read’s cartoon character "PIX" used to illustrate Mason et al. (1982), see footnote 29.
[31] ˚Allusion to Caleb Gattegno.

notice more and more relevant *patterns*.[32] This was supposed to be achieved by gliding up and down a helix, *manipulating* objects, *getting a sense of* some properties of these objects, *articulating* these properties as an expression of generality, then repeating the journey starting from manipulating the more general objects.[33] Calebus explained to the visitors that members of the club thus practiced not only their subtraction skills but also their *abstraction* skills.

And then, suddenly, through *a delicate shift of attention*, Calebus noticed that Odysseus was nothing but a mathematical problem. He immediately *got involved in thinking* about him. With much *grumbling about how things are, griping about specific frustrations, groping for some alternative, grasping at some possibility, grappling with a possible solution and gripping hard to something that seemed to work*[34], he finally got what looked like a plausible answer, without using a single letter symbol in his solution. This put him in a very positive emotional state. Odysseus felt very happy, too. They celebrated the occasion with a nice cup of tea, while enjoying the following conversation.

You are old, dear Calebus[35], Odysseus said,
And your sight has become very weak.
Yet you incessantly notice things in your head.
Could you tell me how you make the link?

To keep my noticing as sharp as a dart,
Calebus replied in his wisdom,
I made sure not to read the great Descartes,
And yet not to abuse of my reason[36].

Walking back to their ship, Odysseus and his *godlike men* passed by another club where, obviously, some important event was about to take place. A group of delegates of the club were waving a banner with "Theorists of the world, unite!" written on it. As one of the participants of the meeting informed Odysseus, the group was expecting the imminent arrival of a guest of honor, a disciple of the famous psychologist Davydus[37]. He was supposed to present a revised version of his theory. The members of the club were all excited about it because they expected to transform their

[32] Mason, J. (1989): Mathematical abstraction as the result of a delicate shift of attention. — In: For the Learning of Mathematics 9 (No. 2), 2-8
[33] ibid.
[34] Mason, J. (1998): Researching from the inside in mathematics education. — In A. Sierpinska & J. Kilpatrick (Eds.), Mathematics Education as a Research Domain: A Search for Identity. An ICMI Study. Dortrecht: Kluwer Academic Publishers, p. 357-377
[35] Paraphrase of the poem "You are old, Father William" from *Alice in Wonderland*.
[36] Allusion to the contrast between Cartesian rationalism and Baconian empiricism.
[37] Davydov, V.V.: (1990): Types of Generalization in Instruction: Logical and Psychological Problems in the Structuring of School Curricula. - In: J. Kilpatrick (Ed.), Soviet Studies in Mathematics Education, Vol. 2. Reston, VA: NCTM, Inc.

Zones of Proximal Development[38], already slightly worn out, into real *Symbolic Spaces*.[39] These spaces, it was said, could help the collective of the club to collectively solve certain mathematical problems. For the time being, however, the individual members of the club, as cognitive subjects, hadn't noticed that Odysseus was a mathematical problem to be solved. Odysseus and his men took advantage of this situation and managed to escape without getting arrested.

5. Crossing the ocean

Well rested after the visit in Calebus' Club, Odysseus embarked on what was to become a long and adventurous journey across the waves of the Ocean. He managed not to totally surrender to the charms of the *mighty* sorceress Technology, he steered a dangerous course between the Scylla of Practice and the Charybdis of Theory, and he had to be bound to his mast not to become enchanted by the *lucid song* of Discourse Analysis Sirens. *Worn-out, in need of sleep*, he reached the *sun-god's lovely island*, whose inhabitants peacefully weaved teaching materials such as software and textbooks. Prophecies advised him to *shun the land of Helios*, but his companions wouldn't listen to his warnings. They decided to nevertheless go ashore, while he was asleep. As expected, they never returned, seduced by a way of life which was, if not easy, then at least less risky and stressful than life on the uncharted waters of research. Odysseus, thus deserted by his crew, continued the voyage alone. Zeus, angered by Odysseus' inability to keep his companions engaged in research, sent him a terrible storm, *striking his ship with blazing lightning, and tearing her to bits upon the wine dark sea*. But the gray-eyed goddess Athena took pity of him and brought him safe ashore.

6. Odysseus in the land of Wagons

Now Dawn, the flowered one, was quick to come. Before Odysseus' eyes, there spread a vast land inhabited by a population of fast-fed mortals. It was a busy morning. Magnificent band*wagons* advanced in front of Odysseus — *high, with sturdy wheels* — each carrying a banner announcing the advent of Understanding in mathematics classes.[40] Odysseus was ecstatic: this is what he had always dreamed about! As he got used to the light, he noticed that there were smaller wagons, too, moving in the background. Their inscriptions were slightly faded out, but he could read, *Problem Solving, Constructivism, Situated Cognition, Communication in the classroom, Ethnomathematics, Language, Discourse analysis*, and others. Suddenly, he saw a group of four chariots detach themselves from "Constructivism" and become very noisy. The chariots bore these letters A, P, O, S, and the leader of

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[38] Vygotsky, L.S.: (1987): The Collected works of L.S. Vygotsky, Volume 1, Problems of General Psychology, Including the volume *Thinking and Speech.* Plenum Press, New York, p. 209

[39] Meira, L., Lerman, S. (1999): *The Zone of Proximal Development as a Symbolic Space*. Manuscript

[40] Fennema, E. & Romberg, T. (Eds.) (1999): Mathematics Classrooms that Promote Understanding. - Mahwah, N.J.: Lawrence Erlbaum Associates, Publishers

each wore a cap of the same form by different color, bright red, green, yellow, blue.[41]. The group chanted, "Action, Process, Object, Schema! Action, Process, Object, Schema, again, and again, and again!" and yelled, "Long live Piaget!" with a lot of vim and vigor. The group of "Situated Cognition" responded by "Down with Piaget!" with much conviction but a lot less enthusiasm.

Intrigued, Odysseus approached the APOS group. He jumped into the first wagon and asked, "what makes you so happy?" The Red Hat replied in these words:

Red Hat: It’s because stand fast despite all the changes. After all, our theory is a theory of understanding in mathematics. A cognitive schema is a basic component of understanding.[42]

Odysseus: What cognitive schemas could a student use to solve me, according to you?

Red Hat: UmÉI never thought about that. You are a word problem, while I have never been interested in problem solving; I focused on concept acquisition and on advanced mathematical concepts, not on elementary ones such as those one needs in your case. But let me try, anyway. One could use fractions and proportional reasoning and perhaps draw a little diagramÉ But, the "fractions" and "proportional reasoning" could be used at the level of interiorized actions or, at most, processes, without these being encapsulated into mental objects and schemas that correspond to what we call, in mathematics, rational numbers and linear transformations.[43].

Odysseus (*astonished*): It is for the third time during my voyage that I hear the word "object", and each time the meaning seems to be different. At the Belvedere, an "object" was an element of a culture or an institution; it was "an object of knowledge". Here, "object" seems to refer to a cognitive structure, or a qualitative and psychological model of the functioning of the mind, and not to a mathematical model of a possible solution of a problem such the "bilinear multiplicative structure" of this colleague from Conceptual Fields.

Red Hat: Contrary to your friends from Belvedere, I think that, in fact, knowledge and its acquisition, or epistemology and psychology are not easily separable.[44].

Odysseus: Thank you for your explanations. There is one more thing that bothers me: can you tell me why, in this country, Understanding in mathematics classes gets so much attention and publicity?

Red Hat: Well, Understanding is the bone of contention in the "math wars" fought in the country. Some camps consider all this insistence on understanding as the Trojan horse that is sure to infiltrate

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[41] °Allusion to the multicolored caps of Ed Dubinsky.

[42] °Rumelhart D.E. (1980): Schemata: The Building Blocks of Cognition. — In: R.J. Spiro, B.C. Bruce, W.F. Brewer (Eds.), Theoretical Issues in Reading Comprehension. Perspectives from Cognitive Psychology, Linguistics, Artificial Intelligence, and Education. Hillsdale, New Jersey: Lawrence Erlbaum Associates, Publishers, p. 33-58

[43] °Czarnocha, B., Dubinsky, E., Prabhu, V., Vidakovic, D., (1999): One Theoretical Perspective in Undergraduate Mathematics Education Research. A Research Forum Presentation. — In: O. Zaslavsky (Ed.), Proceedings of the 23rd Conference of the International Group for the Psychology of Mathematics Education, Vol. I. Haifa, July 25-30, 1999, p. 95-110

[44] °Dubinsky, E. (1991): Constructive Aspects of Reflective Abstraction in Advanced Mathematics. — In: L.P. Steffe (Ed.), Epistemological Foundations of Mathematical Experience. New York: Springer Verlag, p. 160-202

the vulnerable brains of the nations' children with "fuzzy math". These people fight for a return of the good old training for fast and accurate computational skills in mathematics classes.[45] Algebraidon is not sure which side to take, so, for the time being, he is employed by both camps to do their tax returns.

These were his words. Upon learning that Algebraidon is in the country, Odysseus decided to take off and return to his homeland. Knowing that Odysseus has no money to pay his travel, Red Hat helped him to obtain a research grant. With these generous funds, Odysseus constructed a spacious vessel and sailed back home.

7. Homecoming

Back home, Odysseus went to see his old companion *Eumaeus*.

Eumaeus: Things are not going well here, my friend. As usual, nobody cares about human misery in general and about children's difficulties in mathematics in particular. But before, at least the gifted children were receiving some attention, there were methods of identifying them and they were helped in pursuing mathematical studies. In fact, these children were treated as little princes.

Odysseus: It's better like this, perhaps. *Batiushka* Tolstoy wouldn't have liked this discriminatory attitude; he advocated education for all people. But tell me, pray, how can you know if someone is or is not mathematically gifted? Those in the business of selection, did they have a definition?

Eumaeus: Of course they had a definition! These scientists were capable of anything! All school children knew it by hearth, just like the *Internationale*. The definition was saying that is gifted in mathematics someone who can formalize, generalize, symbolize, visualize and reason logically, economically, directly as well as indirectly, flexibly and without being overly influenced by the ordinary sense of the words and common sense habits.[46] You can see that this is not easy.

Odysseus: But how did the specialists go about diagnosing a mathematical talent?

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[45] For more about "math wars" in the US, see Kilpatrick, J. (2001): Understanding Mathematical Literacy: The Contribution of Research. — In: Educational Studies in Mathematics 47, p. 101-116.

[46]°More precisely, a mathematically gifted child is inclined to:
(a) formalize the mathematical material, distinguish its form from its contents, abstract from the concrete numerical and spatial relations and operate with a formal structure of relations;
(b) generalize the mathematical material and remember these generalizations;
(c) operate with symbolic representations of numbers, relations and other mathematical entities;
(d) reason logically;
(e) take mental shortcuts while solving problems;
(f) move easily from a direct to an inverse mode of reasoning; in particular — from a direct proof to a prof by contradiction, and from a theorem to its converse;
(g) pass easily from one mental operation to another and not allow oneself to be overly influenced by the vernacular meanings of words or common sense habits;
(h) visualize spatial relations
(based on: Krutetskii, V.A. (1976): The Psychology of Mathematical Abilities in Schoolchildren. - Chicago & London: The University of Chicago Press, p. 84-88).

Eumaeus: For each characteristic of a mathematical talent they had a set of systems of problems taken in different domains of mathematics, such as arithmetic, algebra, geometry. One system was made of 5 to 6 problems and, each of these problems was written in 4 to 5 versions, coded (a) to (d) or (e). Let me give you an example. Suppose you want to diagnose the ability for abstract thought in children. So you invent a set of systems of problems. In the domain of arithmetic and algebra, let's take (to make it simple) a series of, say, three problems. The problems go from simple to more complicated and their versions, say, (a), (b), (c), (d) are more and more demanding with respect to abstract thinking. The interview with a child starts with the first (the simplest) problem in its version (d), i.e. the most demanding with respect to abstract thinking. If the child solves the problem, he or she gets the second problem to solve, again in the most abstract version, i.e. 2(d). If the child fails to solve the problem 1(d), he or she gets the problem 1(a), or the least demanding from the point of view of abstraction. You count the number of versions the child needed to solve before getting to the most demanding one (some children may never get there). At the end you compute a pair of indices, (T, S), where T is the number of problems the child has solved and S is the mean of the number of versions he or she needed to get to the most abstract version.[47]. I'll give you an example of such a series of problems; you will be the last and the most complicated of the series, but also the least demanding version (3a) of this problem with respect to abstract thinking.

Problem 1

1a. A carpenter and his apprentice prepare a plank of wood for the roof of a house in 1 hour. The apprentice is two times slower than the carpenter in doing this type of work. How much time would the apprentice need to prepare the plank alone?

1b. A carpenter and his apprentice prepare a plank of wood for the roof of a house in 1 hour. The apprentice is three times slower than the carpenter in doing this type of work. How much time would the apprentice need to prepare the plank alone?

1c. A carpenter and his apprentice prepare a plank of wood for the roof of a house in 1 hour. The apprentice is one and a half times slower than the carpenter in doing this type of work. How much time would the apprentice need to prepare the plank alone?

1d. A carpenter and his apprentice prepare a plank of wood for the roof of a house in 1 hour. The apprentice is a times slower than the carpenter in doing this type of work. How much time would the apprentice need to prepare the plank alone?

Problem 2

2a. Ivanushka and Verochka used to go up the hill to fetch some pails of water and fill a tank in their farmyard. They normally needed two hours to bring 30 pails of water, which filled 1/2 of the tank. One day, Ivanushka fell down and broke his crown and Verochka had to do this work alone. It took her 4 hours to bring 30 pails of water. How much time would it take Ivanushka to fill the tank, if the accident happened to Verochka and not to him?

[47] °Krutetskii, ibid., p. 125.

2b. Ivanushka and Verochka used to go up the hill to fetch some pails of water and fill a tank in their farmyard. They normally needed two hours to bring 30 pails of water which filled 2/3 of the tank. One day, Ivanushka fell down and broke his crown and Verochka had to do this work alone. It took her 5 hours to fill the tank. How much time would it take Ivanushka to fill 1/3 of the tank?

2c. Ivanushka and Verochka used to go up the hill to fetch some pails of water and fill a tank in their farmyard. They normally needed two hours to bring *a* pails of water which filled a fraction *b* of the tank. One day, Ivanoushka fell down and broke his crown and Verochka had to do this work alone. It took her 5 hours to fill the tank. How much time would it take Ivanushka to fill the fraction *c* of the tank?

2d. Ivanushka and Verochka used to go up the hill to fetch some pails of water and fill a tank in their farmyard. They normally needed *d* hours to bring *a* pails of water which filled a fraction *b* of the tank. One day, Ivanushka fell down and broke his crown and Verochka had to do this work alone. It took her *e* hours to fill the tank. How much time would it take Ivanushka to fill the fraction *c* of the tank?

Problem 3

3a. A team of reapers was given the task of scything two meadows, one of which was the double the size of the other. One half of a day the team worked on the larger meadow. For the second half of the day the team separated into two equal groups. One of the groups stayed in the large meadow and finished the work by the evening. The other group started mowing the smaller meadow, but, by the end of the day, there was still a part of the work to do. This part was done the next day by one man in one day's work. How many reapers were there in the team?

3b. A team of reapers was given the task of scything two meadows, one of which was 4/5 the size of the other. One third of a day the team worked on the larger meadow. For the rest of the day the team separated into two groups one of which was twice the size of the other. The larger group stayed in the bigger meadow and finished the work by the evening. The smaller group started mowing the smaller meadow, but, by the end of the day, there was still a part of the work to do. This part was done the next day by two men in one day's work. How many reapers were there in the team?

3c. A team of reapers was given the task of scything two meadows, one of which was p/q the size of the other (p, q positive integers, p < q). One third of a day the team worked on the larger meadow. For the rest of the day the team separated into two groups one of which was twice the size of the other. The larger group stayed in the bigger meadow and finished the work by the evening. The smaller group started mowing the smaller meadow, but, by the end of the day, there was still a part of the work to do. This part was done the next day by two men in one day's work. How many reapers were there in the team? For what values of p and q does the problem make sense?

3d. A team of reapers was given the task of scything two meadows, the smaller one of which was *a* times the size of the larger one (a rational, $0 < a < 1$). During part *b* of the day (b rational, $0 < b < 1$) the team worked on the larger meadow. For the rest of the day the team separated into two groups one of which was c times the size of the other (c positive integer). The larger group stayed in the bigger meadow and finished the work by the evening. The smaller group started mowing the smaller meadow, but, by the end of the day, there was still a part of the work to do. This part was done the next day by k men in one day's work. How many reapers were there in the team? For what values of the variables does the problem make sense?

Odysseus was growing more and more impatient with each new version of each new problem. He sighed, "O Zeus, I see that your brother Algebraidon has not relented in his fury. Wherever I go he turns me into an algebra problem!" Eumaeus continued his presentation: Suppose these problems were given to three students, Potapov, Nikolskij and Faddeev, and the results were coded (1˚;˚4), (2˚;˚3) et (3˚;˚2) respectively, on a maximum of (3˚;˚1). This means that Potapov solved only one of the problems in version (d) and had to go through all 4 versions to get to the most demanding from the point of view of abstract thinking. Nikolskij did 2 problems in version (d) and needed to solve, on the average, three versions to get to the most demanding one. Faddeev did all three problems in version (d) and needed to go through only 2 versions, on the average, to solve the most demanding one.

Odysseus tried to settle in his native land but was quickly discouraged. People were no longer interested in solving arithmetical riddles. They had more serious problems to cope with. He started a business but Algebraidon got mixed up in his bank accounts and the balance started showing negative integers. This was too much for Odysseus, for whom these numbers were not good even as "useful fictions". He decided to leave.

8. ODYSSEUS IN THE LAND OF PARTIAL DIFFERENTIAL SOLIDARITY

When Dawn's rose fingers reached the sky, Odysseus woke up in a land known for the solidarity of its inhabitants and large differences of opinion. He tried to travel incognito, but was immediately recognized. Textbook publishers, desperate for interesting problems, surrounded him. He was offered good salaries, but, everywhere, it was the same condition: he had to dress up properly. His casual clothes were no longer acceptable. One publisher gave him this nice suit:

Mr Alepski is the milk supplier for *Danone*. He has a herd of cows and needs hay to feed them. But hay must first be reaped. This is not easy because all his workers are presently busy negotiating agricultural subventions with the government by means of road blockages. It is absolutely necessary that his two meadows, one of which is twice as large as the other, be reaped in at most two days. He decides to negotiate with the farmers the return to work of at least a few of them. Suppose that, one half of a day the team would work on the larger meadow. For the second half of the day the team would separate into two equal groups. One of the groups would stay in the large meadow and finish the work by the evening. The other group would start mowing the smaller meadow, but, by the end of the day, there would still remain a part of the work to do. This part could be done the next day by one man in one day's work. Help Mr Alepski to calculate the minimal number of workers he will have to negotiate.

But Odysseus was not very comfortable in this rigid attire. He loved the great outdoors too much. He decided to go cod fishing. He sailed the Ocean again. One day, stormy weather brought his ship to the integral shores of an internally divided country.

9. ODYSSEUS IN THE LAND OF SEPARABLE VARIABLES

When Dawn's rose fingers reached the sky, Odysseus came to a vast tower filled with people speaking a multitude of languages. They were participants in congress whose aim was to bring together two communities, which, far from considering language as a variable property of speech, insisted, on reducing it to a single value. The problem was that this value was different for each community, which made the communication somewhat difficult but all the more full of meaning. There was a lot of good will on both sides and curiosity was drawing people together.

Odysseus went to a working group session. He tried to just sit quietly in a corner and watch the proceedings, but this turned out to be impossible in the culture of this land. He was called forth to share his ideas, however half-baked, with the others, and work on approximate solutions to precise questions in small groups. But it was very difficult to work! Not only each participant seemed to speak a different language; he or she also seemed to think about something different and tried to solve a different problem. Finally, the only thing that could be done in common was to chat / *causer*. Slowly, however, he got used to the situation and even started to enjoy it. *CÕtait lÕun* to see oneself being understood in so many different ways! It was like living several lives at once.

He relaxed in the conversations. Presently a didactician was reflecting on the possibility of organizing a didactic milieu around the problem of reapers in the aim of engineering such structural changes in students' ways of knowing that would allow them to pass from arithmetical to algebraic thinking.[48] An educator rejected this idea, horrified by the underlying "epistemological determinism".[49] He argued that it was not possible to engineer a pre-determined cognition just as it was impossible to engineer a predetermined natural selection; it was a contradiction of terms! Both processes are biological — he explained. The didactician didn't agree: "But didactic engineering has nothing to do with cognitive engineering! You don't understand what I mean!". The educator retorted by saying that didactic engineering can only lead to deception because the educational system is an autopo etic system[50]: all we do to, supposedly, change it, only creates the condition for wanting to change it again. In fact, the state of permanent reform is a distinctive characteristic of the organization of the educational system.

At this point, some participants panicked: "That's all very interesting, but the awareness of this phenomenon may lead to meekness and despondency. Why do anything, if all one gets is a reproduction of the system?"

The debate became so heated that its fire burned down the importance of both Odysseus, the little arithmetical problem, and of his tormenter, God Algebraidon. This common misfortune brought the two foes together. Hand in hand, they went off to merge in the Theory of Fields, thus clearing the discussion of all its mathematical contents.

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[48] Bednarz, N., Janvier, B., Mary, C. (1992): LÕAlg bre comme outil de r solution de probl mes°: une r flexion sur les changements n c ssaires dans le passage dÕun mode de traitement arithm tique un mode de traitement alg brique. - In: Recueil des Textes du Colloque du Programme de Recherche sur lÕmer gence de lÕAlg bre, C.I.R.A.D.E., UQAM, le 10 avril 1992, p. 3-15

[49]°In the sense of the possibility of predicting future cognitive acts. This is a reference to Maturana & Varela (see note 43 for reference) where a distinction is made between "determinism" and predictability.

[50]°Maturana, H.R., Varela, F.J. (1987): The Tree of Knowledge: The Biological Roots of Human Understanding. - Boston & London: New Science Library, p. 43