LECTURE 7

DIDACTIC ENGINEERING 2:

CAUSES OF DIFFICULTIES OF REALIZATION OF DIDACTIC ENGINEERING PROJECTS.

It is often the case that a teaching project which, a priori, appears to create the appropriate conditions for the construction, by the students, of a given mathematical knowledge, is disappointing when it is experimented. Students either do not learn anything new or learn something different from the target knowledge.

What could be the objective[1] causes of this failure? We shall discuss three types of causes, labeled '*didactic transposition'*, '*teacher's epistemology'*, and '*obstacles*' of various origins. It may also happen that a teaching experiment 'works' once, but the success is not repeated when it is run for a second time. Why? This is the problem of *reproducibility* of the products of didactic engineering.

1. DIDACTIC TRANSPOSITION

The didactic engineering methodology presupposes that, at least in the phases of action, formulation and validation, the students will act not as students whose sole aim is to satisfy the teacher and pass the course, but as learners whose aim is to gain some new mathematical knowledge by solving a problem. It is implicitly expected that, at some point, the students will behave as mathematicians.

This expectation can be justified only if the target knowledge is, indeed, of the same nature as mathematicians' knowledge. But this need not be the case.

In general, school mathematics knowledge differs quite substantially from mathematicians' knowledge, not only quantitatively: school mathematics knowledge is not a small subset of the mathematician's knowledge. The school institution has developed mathematics as a teaching subject which may have parts not included in the mathematicians' knowledge.

One could say that, in general, mathematical knowledge (or any kind of knowledge for that matter) is a function of the institution in which it 'lives'. The mathematics developed and

[1] 'Objective' here means: independent from the personal characteristics of the teacher and the students (e.g. poor class management of the former and the incurable laziness of the latter).

used in an Arts and Science Department is different from the mathematics developed and used in the Engineering Department, which is yet different from the mathematics as it serves the medical and biology research centers. There is Actuarial Mathematics and Financial Mathematics. There is not one 'La Mathématique', as it was held in the 1960s, during the New Math reforms. Of course, all these various institutions are not hermetically closed and there is a continual flow of ideas between them. When an institution picks up some body of knowledge from another one and adapts it, changes it to fit her own goals and tasks, then we speak of an '*institutional transposition of knowledge*'[2]. When the transposition goes from an institution which produces knowledge to an institution which teaches it to students, then we speak of '*didactic transposition*'[3].

The process of didactic transposition is inevitable when it comes to teaching mathematics at school; however teachers and curriculum developers must beware of producing knowledge that would have only some kind of 'internal' value for the functioning of the school as an institution, but not outside of it. We must constantly remind ourselves that the aim of the school is to prepare the children for life and professions outside of the school. The risk is real; we have seen monstruous examples of such unnecessary 'didactic creativity' in mathematics teaching, such as the 'language of strings and arrows', in the 1960s New Math reforms. There are less monstruous examples such as the 'Big cosine', Cos and the 'Small cosine', cos.

Exercise 1

Can you give other examples of school mathematical concepts which do not exist in the academic mathematics?

Aside from creating new 'objects of knowledge', school mathematics differs from 'research mathematics' also in other ways.

A part of some general knowledge can be isolated and given an important status of a 'topic' with a special name and place in the curriculum. Example: 'les identités remarquables' in the French curriculum. This refers to algebraic properties of operations on real numbers which are, in the theory, simple consequences of the properties of the real number system

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[2] Chevallard, Y. 1992: Fundamental concepts in didactics: perspectives provided by an anthropological approach. In R. Douady & A. Mercier (eds.), *Research in Didactique of Mathematics. Selected Papers*. Grenoble: La Pensée Sauvage éditions, pp. 131-167, p. 165.

[3] Actually, in participating in the present course, you are witness of a didactic transposition of a body of knowledge that has first evolved as a university research domain, the 'Theory of Didactic Situations'. I shall not dwell on the differences between this academic body of knowledge and the knowledge that we are constructing together in the course, but I am fully aware of the existence of such differences.

(commutativity and associativity of addition and multiplication, distributivity of addition with respect to multiplication, etc.), e.g. $(a-b)(a+b)=a^2 - b^2$.[4] The curriculum devotes several class periods for this topic and students are given many exercises for their application; a special test is usually designed for the assessment of the students' mastery of these identities.

A technique that mathematicians are using as an instrument and are not studying per se becomes an object of teaching: this is the case of 'paramathematical notions' such as equation, parameter or proof[5]. A mathematician would not say: 'today, I've been doing proofs', or 'I've been solving equations with parameters' because he or she is doing proofs every day or every time he or she is doing mathematics, and in all his problems there are constants, variables and parameters. But a teacher can be 'doing proofs', or teaching the 'notion of equation', or 'the notion of parameter' to his or her students today and consider the topic finished and done with tomorrow[6].

Perhaps the most important difference between research mathematics and school mathematics is the aim of the mathematical activity. The mathematician wants to know and in the aim of knowing he uses his or her mathematicial competences; the student has to demonstrate that he or she knows what he or she is expected to know and possesses the competences aimed at by the curriculum. For example, a mathematician will transform an expression like $4x^2 - 36x$ into $4x(x-9)$ or $(2x-6\sqrt{x})(2x+6\sqrt{x})$ depending on the assumptions and purposes of the problem in which this expression appeared. But the student is quite likely to receive this expression isolated from any mathematical problem, in an exercise like 'Factor $4x^2 - 36x$' and he or she will know to be expected to answer with $4x(x-9)$ or $(2x-6\sqrt{x})(2x+6\sqrt{x})$ depending on the didactic, not mathematical context of this exercise, i.e. on what the teacher had been teaching before giving this exercise. So, in choosing what to write the student is not so much solving a mathematical problem as deciphering the rules of the didactic contract[7].

A thorough analysis of the distinction between school and research mathematics in general is available in Chevallard (1991) (see footnotes). An in-depth study of the didactic

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[4] Chevallard, Y. 1991: *La transposition didactique du savoir savant au savoir enseigné*. Grenoble: La Pensée Sauvage éditions, p. 43.
[5] Proviso: the 'notion of proof' is an object of study for logicians working in the so-called 'foundations of mathematics'. But even for them mathematical proof is an everyday tool; they justify their assertions by means of proofs.
[6] Chevallard, 1991, p. 49-51.
[7] ibid., p. 52.

transposition of a particular mathematical concept, the concept of distance, can be found in Chevallard & Johsua (1991)[8].

2. TEACHER'S EPISTEMOLOGY

Knowledge functions in the school institution in a very different way than in a research institution. It may therefore not be very realistic to expect that, even through the best - a priori - didactic engineering, students will develop mathematical ways of thinking and concepts similar to those of research mathematicians.

The way mathematical knowledge functions at school is influenced by the teacher's knowledge relative to the acquisition of mathematical knowledge. It is not necessary that the teacher be aware of that knowledge, that he be able to verbalize it and discuss it against other possible views on the acquisition of mathematical knowledge. The existence of this knowledge manifests itself in the teacher's practices. Brousseau calls this knowledge 'teacher's epistemology' (p. 35)[9].

Here are some examples of such manifestations.

1. A teacher will classify some students' errors as 'une étourderie' (a result of absent-mindedness), especially if they come as an answer to an 'easy' question that most students already are familiar with. They will classify some other errors as 'conceptual' or simply 'serious'. The former will not entail any didactic action on their part; the latter will - the teacher will engage in 'revisions', 'remedial activities', etc., with respect to students whom they suspect of having 'conceptual problems'. This is a symptom of some kind of 'spontaneous psychology' that the teacher seems to profess, which has little to do with research in cognitive psychology, where responses to cognitive problems are not classified into 'serious' and 'not serious'.

2. The teacher expects the students to produce their answers according to a schema which they consider as 'correct'; they expect the students to produce their answers in an 'intelligent' way, acceptable within the mathematical culture they believe the school should represent. Some teachers thus confuse the laws of the production of knowledge with the systematization and organization of knowledge. But even those who know that, in the domain of real mathematical research, solutions to problems are not found this way or that, in general, human cognition does not work in such a systematic, orderly and logical fashion, will still hold such expectations and

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[8] Chevallard, Y. & Johsua, M.-A. 1991: *Un exemple de transposition didactique*. Grenoble: La Pensée Sauvage.

[9] The explanations of the phenomenon of 'teachers' epistemology' that you will find in this section are a free translation and adaptation of a part of Brousseau's clarifications which I obtained from him in private e-mail correspondence.

requirements with respect to the students' answers: this is a necessary implication of the didactic contract. The teacher cannot transform his or her class in a psychology lab, and he or she cannot straighten up all the individual or even collective trajectories of learning because this would be too time- and labor consuming and perhaps technically impossible.

Thus, in his or her practice the teacher uses certain 'concepts' or 'laws' related to epistemological questions such as 'what does it mean to investigate?', 'to learn?', 'to understand?' which define the students' field for action and justify the teacher's decisions. This spontaneous epistemology is accompanied by a whole mythology of metaphors and symbols thus forming a system - a praxis of the teacher - which makes his work appear possible and legitimate.

Here are some more examples of teachers' decisions that reflect this spontaneous epistemology and show how distinct it can be from an epistemology actually governing the work of mathematicians.

In mathematics it does not matter very much how a result has been established: whether by way of a witty reasoning or by way of a laborious calculation and verification of all possible cases. What matters is to show that the solution solves the problem. But if a student solves an equation by trying some numbers at random or even systematically, his solution will not be considered correct by the teacher. It is worthless from the point of view of the teacher's epistemology, because it has no 'positive *didactic* value': the student has not demonstrated he or she had learned the method. The aim, at school, is not to solve the problem but to demonstrate one has learned what one has been taught[10].

Teachers' spontaneous epistemology of mathematics is often at odds with the foundations of the theory of situations. Since it is this theory that underlies teaching projects conducted according to the methodology of didactic engineering, this discrepancy may explain the difficulties of the classroom realization of these projects.

Exercise 2.

In an article published in English in the journal For the Learning of Mathematics in 1991, Michèle Artigue and Marie-Jeanne Perrin-Glorian[11] report about a didactic engineering experience in a middle school (10-12 years old children) with students who had difficulties in mathematics (and other subjects). It turned out to be practically impossible to realize the didactic engineering projects, and the researchers started to study the reasons for this failure. They have studied, in particular, the teachers' conceptions related to the teaching and learning of mathematics.

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[10] This is the end of Brousseau's explanations.

[11] Artigue, M. & Perrin-Glorian, M.-J. 1991: Didactic Engineering, Research and Development Tool: Some Theoretical Problems Linked to this Duality. *For the Learning of Mathematics 11*.1, 13-18.

Based on your reading of this part of the article, explain in what way could these conceptions contribute to the failure of the didactic engineering? In what way are these conceptions incompatible with the theory of situations?

3. Reproducibility of didactic situations

It has proved extremely hard for teachers to reproduce a didactic situation so that the mathematical meaning of the students actions is conserved. This phenomenon is called, by Brousseau, the 'obsolescence of didactic situations' (p. 193).

By obsolescence we mean the following phenomenon: from one year to another, teachers have more and more trouble reproducing the conditions likely to lead their students to create, perhaps through different reactions, the same understanding of the notion taught. Instead of reproducing conditions which, while producing the same result leave the trajectories free, they reproduce, on the contrary, a "history", a development similar to that of previous years, by means of interventions that, even if discrete, completely change the nature of the didactic conditions guaranteeing a correct meaning for the students' reactions; the obtained behavior is apparently the same but the conditions under which it was obtained modify the meaning (p. 193).

The failure of even the first realization of a didactic engineering project is related to this difficulty in reproducing the mathematical meanings. In writing up a scenario for a didactic situation, it is rather easy to describe the physical environment, the tasks, the verbal interventions and the actions of the teacher, the expected reactions of the students. It is much more difficult to enumerate and highlight those features of the milieu which are indeed responsible for the emergence of the target mathematical meaning. Sometimes it can even be hard to tell what these features are, and it may well be that the dynamics of a didactic situation represent a chaotic rather than a stable system, meaning that a tiny alteration of its conditions (e.g. an apparently unimportant remark or a hint from the teacher, an expression of approval or disapproval on his or her face) may cause huge changes in the knowledge that it produces[12].

In order to increase the chances of reproducibility, the didactic engineering methodology is very demanding with regard to the a priori analysis of the target knowledge and the work of identifying the critical variables of the didactic situations that are supposed to lead to it.

Example.

A group of researchers[13] have studied the problems of the reproducibility of the products of didactic engineering in the following setting: The same, a priori, didactic situation has been implemented by two different teachers in two different classes of 13-14 years old students. The

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[12] ibid., p. 14.

[13] Arsac, G., Balacheff, N., Mante, M. 1992: Teacher's Role and Reproducibility of Didactic Situations. *Educational Studies in Mathematics 23*, 5-29.

mathematical content of the situation was the activity of proving. The situation was composed of two main phases:

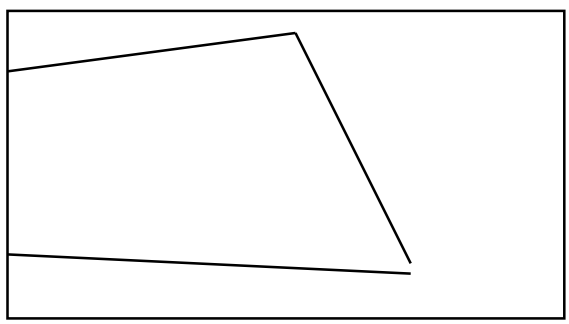
(a) the research phase, in which the students are given a problem and write their solutions on a poster; the problem is so chosen that there are many solutions possible, and it is likely that the students will disagree on many points;

(b) the debate phase: the students' solutions are written on large sheets of paper and are then displayed as posters on the walls of the classroom. Students are divided into teams which analyze the solutions. Each team then delegates one spokesperson who presents the result of the analysis to the whole class. A whole class debate is then engaged.

The situation was planned to last over two one and a half hour periods.

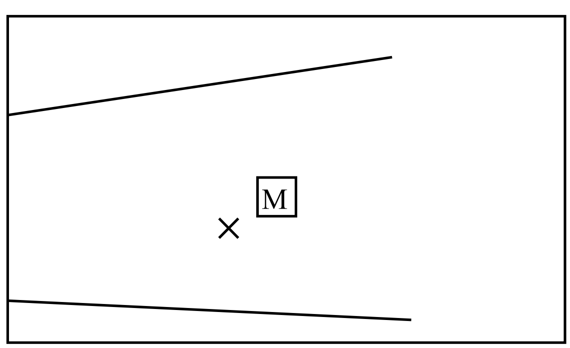
In Class I, the teacher was a member of the research team. In Class II, the teacher was not a researcher but the scenario was thoroughly presented and discussed with her.

In Class I the problem given to the students was the following:



'*Write for other students a message allowing them to come to know the perimeter of any triangle a piece of which is missing. To do so, your colleagues will have at their disposal only the paper on which is drawn a triangle and the same instruments as you have (rulers, etc.).*'

In Class II the problem given to the students was the following:



'*The two lines intersect outside a page. Write a method allowing anybody to draw the line through M and the point of intersection without going out of the page*.'

The teacher was supposed not to interfere at the mathematical level during the whole activity. The students had to feel that it is entirely their reponsibility to decide which solutions are valid and which are not. Otherwise, the researchers claimed, the real activity of mathematical proving will not emerge, and the students will only want to produce statements acceptable by the teacher and the teacher's authority will be the ultimate criterion of validity of solutions.

The teacher was allowed to intervene only at the level of the presentation of the problem and the organization and chairmanship of the debate.

However, in the implementation in Class I, the teacher intervened in ways that did affect the students' attitude towards the mathematical activity and, in actual fact, relieved them from the responsibility for the mathematical validity of their solutions.

Here are some examples of this kind of interventions, as reported by the researchers:

- in order to guarantee that the research phase be not too long, the teacher invited the students to propose a solution as soon as she thought that what they obtained was sufficiently developed, and not when she was sure that the students thought so;
- the teacher was drawing the students' attention to some crucial words in the formulation of the problem (in particular, the word 'any' in Class I);
- the teacher in Class I kept continuous contact with the students, making about one intervention every minute over an 80-minute period (from 'Are you okay?', to 'Have you read the problem carefully?');
- when she considered that a discussion is irrelevant from the mathematical point of view and leads nowhere, she would urge the students to dismiss it and go further; she would also focus the students' attention on solutions and ideas that she considered relevant, and sum them up for the students.

As a result of these interventions the students 'got confused and were no longer committed to any real discovery of the solution'.

In a repetition of the experiment in another class, the teacher was asked explicitly not to intervene at all, and stay at her desk during the whole phase of discovery. She did. She only told the students what is the problem that they have to solve and told them that they have so much time for it. But this did not make the students engage in the activity of proving, either. The students compared the solutions with respect to their simplicity, clarity, usability, but not their validity. When, after one solution was accepted by the whole class, the teacher asked if the authors of this solution are sure of it. They responded: 'Yes, because we have done it in a lot of cases'. So it was not even enough for the teacher to directly ask for a proof, to make the students produce a proof, although, as it was found out, they were perfectly able to prove that their solution was correct, in a mathematical way.

In Class II, the teacher intervened in a stronger way even than the teacher in Class I: she asked leading questions ('Do you all agree? It's not proved'), she did not transcribe some solutions on the big sheets of paper, she intervened directly on the content of the debate, she reinforced some students' ways of proving ('Now you make a drawing in each group'. 'It works, but only if M is on the bisector', 'Look at your drawing and come to an agreement').

The researchers concluded by saying that as soon as the teacher (a) tells the students that they have a limited time to solve a problem, and (b) shows the students that she endorses the 'epistemological responsibility' for the mathematics produced in class (e.g. when she refuses to write a false statement on the board), the students have no reason for entering a genuine activity of proving in the mathematical sense.

It turns out that it is very difficult to implement a teaching design in a way which conserves the intended mathematical meaning of the activities. The meaning of the activity may change from one implementation case to another, due to the teacher's interventions. The teacher's interventions are caused by her beliefs about her role and her duties as a teacher, and when she acts on the basis of these beliefs, she may not support the development of the students' development as autonomous thinkers.

Exercise 3.

Find a few critical features of the didactic situation 'Race to 20' (pp. 3-18) that, if altered, would lead to its failure in attaining such objectives as: discovery and proof, by the children, of a sequence of theorems related to the notion of Euclidean division in integers ('division with a remainder').

4. COGNITIVE, DIDACTIC AND EPISTEMOLOGICAL OBSTACLES

When students start upon learning a new mathematical concept, their minds are not blank slates, they are filled with all kinds of knowledge, beliefs, experience. New knowledge is not simply added on, it must be merged with the old knowledge. But new knowledge can sometimes contradict the old knowledge, and then the old knowledge functions as an obstacle to learning the new one.

The contradictions are inevitable because whatever knowledge we have, is an answer to a domain of questions and problems. This is usually a limited domain, but as long as we have not transgressed its limitations, we believe that there are no limits and our knowledge is universal. So when we find ourselves in front of new questions and problems we want to solve them with the knowledge we have and this knowledge may not be capable of solving these new problems. We need to reject parts of our old knowledge, or re-organize it, or generalize it and recognize that it had a limited domain of application. This process feels as if there was something blocking our mind - an obstacle.

It is not always possible to know in what ways the students' old knowledge can function as an obstacle in learning the knowledge aimed at by a didactic engineering project. But this old knowledge may bias the students' interpretation of the tasks given to them and they may thus completely miss the mathematical point of the activities.

According to Brousseau, obstacles that appear in the teaching of mathematics can be of various origins: ontogenic, didactic, epistemological, cultural (p. 86).

Ontogenic obstacles are ways of knowing whose limitations are due to the stage of the mental development of the child. A 6 year old child cannot be expected to understand the principles of an axiomatic theory.

Didactic obstacles are ways of knowing whose limitations stem from a certain way of teaching. For example, in elementary courses in physics at high school, vector magnitudes such as force or velocity are taught in a context of problems that minimizes the complexity of mathematical computations. If the students have not studied the cosine theorem in mathematics but did study the Pythagorean theorem, composition of magnitudes problems will deal with orthogonal vectors only. The students risk to develop a conception that vector addition applies to orthogonal vectors only, which will function as an obstacle in their study of vector geometry and physics later on. Moreover, school problems related to vector magnitudes in physics tend to concentrate on the measures of the magnitudes; the direction of action of a force or motion is given either implicitly in a diagram accompanying the problem, or in non-mathematical terms such as 'North-West' in the text. Direction is not something that is calculated and it is normally not the object of the question in the problem. So it becomes unimportant. This may explain why in vector geometry, many students believe that a vector can be completely defined by its length[14].

Epistemological obstacles are those whose limitations are related to the very meaning of the mathematical concepts. A mathematical concept has many levels of generality and abstraction, and many aspects, developed during its history and depending on the context of its use. Each level and aspect has its limitations, and if one thinks of a concept in a meaning that is not appropriate for the given context or problem, then this way of thinking functions as an

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[14] Knight, R.D. 1995: The Vector Knowledge of Beginning Physics Students. *The Physics Teacher 33*, pp. 74-78.

obstacle and one makes mistakes or cannot solve the problem. For example, one can think of real numbers as measuring numbers, or as elements of an algebraic structure called a well ordered commutative field. The latter understanding is useless in the context of problems related to the measurement of lengths, areas and volumes of geometric figures. The former is useless when one deals with questions such as: could we put all real numbers in a sequence?[15]

[15] More about epistemological obstacles can be found in:
Sierpinska, A. 1994: Understanding in Mathematics. London: Palmer Press, pp. 112-137.
Sierpinska, A. 1992: On Understanding the Notion of Function. In E. Dubinsky and G. Harel, *The Concept of Function, Aspects of Epistemology and Pedagogy*. MAA Notes, Vol. 25, pp. 23-58.
Sierpinska, A. 1991: Some Remarks on Understanding in Mathematics. *For the Learning of Mathematics 10*(3), 24-26.