LECTURE 5

MORE PHENOMENA OF TEACHING:

THE METACOGNITIVE SHIFT, THE METAMATHEMATICAL SHIFT, THE IMPLICIT SUGGESTION OF ANALOGY, THE PARADOX OF LEARNING BY ADAPTATION, THE PARADOX OF THE ACTOR

This week we continue discussing the phenomena of mathematics teaching as identified by Brousseau.

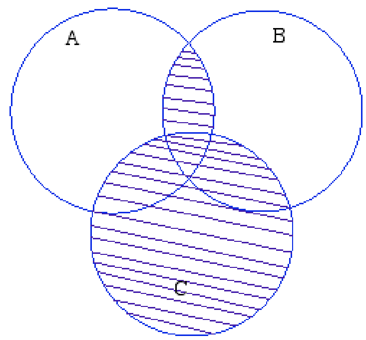
1. THE METACOGNITIVE SHIFT

The phenomenon of the 'metacognitive shift' occurs when a teaching aid becomes an object of teaching itself (p. 26-7).

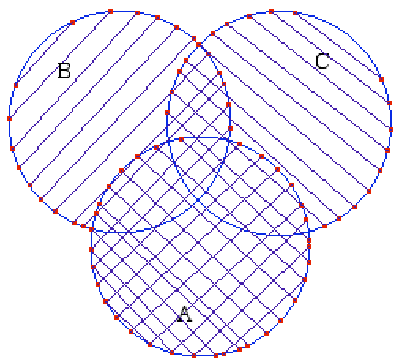
Teachers use all kinds of teaching aids to convey the meaning of the abstract mathematical concepts. They can be material objects such as counters, sticks, blocks, or graphical representations, or orally communicated metaphors. However, the interpretation of actions on and with these objects as representations of the particular mathematical concepts that the teacher has in mind requires that the students focus their attention on certain features of the objects and not on others, and manipulate these objects in some appropriate ways for this particular goal. Otherwise, they may miss the concept completely. If, in order to avoid this, the teacher starts teaching the students the rules of interpreting and using things that were supposed only to help the students grasp the meaning of a mathematical concept, we have to do with the 'metacognitive shift' in teaching.

Example 1: The Venn diagrams

When, during the New Math reform period in the 1960s, elementary school children were taught operations on sets, they could not be expected to reason about them using the language of formal logic. Thus a less formal language was needed, and it was proposed to use the so-called Venn diagrams. A Venn diagram is composed of a number of circles, each representing a set, and the mutual position of the circles is intended to represent the relation between the sets. Thus, for example, the set A□ (B□ C) could be represented by the diagram:



The set $(A \cup B) \cap (A \cup C)$ can be represented by the diagram



Comparing the two diagrams the students would be expected to 'discover' the distributivity law: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

The representation seems to be quite simple and straightforward, as long as it serves not as a formal language but a heuristic tool; something one quickly draws such a diagram on scrap paper to convince oneself about the truth or falsity of a relation between sets, and then writes a proof in ordinary mathematical language (based on the definitions of set operations and the tautologies of the predicate calculus). But, in the teaching of set theory at an elementary level the 'ordinary mathematical language of set theory' was not available. So there was no alternative language to communicate and test the validity of statements: there was only the 'language of strings'[1] or 'Venn diagrams'. To smooth out communication between the teacher and the students, and avoid misunderstandings, the ad hoc drawings for the personal use of mathematicians had to be developed into a language. And this is indeed what we can observe in the abundant literature for teachers produced at the time of the reforms. For example - in the 'Unified Modern Mathematics' series (1972) produced for the use of Teachers College, Columbia University by an impressive international board of mathematics educators called 'Secondary School Mathematics Curriculum Improvement Study' (including, beside the American teacher educators, such 'big names' as Gustave Choquet from France, Lennart Råde from Sweden, and Hans-Georg Steiner from Germany). The Volume 2.1 of this series contains 6 pages of all kinds of rules concerning the interpretation of Venn diagrams. In particular, it is proposed to write the symbol ø in a region of a Venn diagram to mark that it represents an empty set and the symbol 'x' to mark that the region is not empty (ibid., p. 14)[2]. In this approach, the students were given exercises just for the practice of the conventions of the representation (ibid., p. 19). At the time of the New Math reforms, Venn diagrams were used everywhere, but in some places this representation received more attention than in others. In Europe the most famous advocate of the New Math reforms who contributed a lot to the spreading of the use of the 'language' of Venn diagrams and arrow diagrams (for relations and functions) was certainly the Belgian mathematician Papy. He added color coding to his Venn diagrams and represented empty sets not by the symbol ø but by hatching the regions of the diagram[3].

When presenting the phenomenon of the 'metacognitive shift', Brousseau gives the example of the abuse of the Venn diagrams used in teaching set algebra and arrow diagrams in teaching about relations and functions, but can we think of other examples, not necessarily coming from the history of the New Math reforms but from the present day mathematics teaching? I have thought of two examples; can you come up with more?

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[1] In some American textbooks for teachers, the expressions 'the language of strings' rather thah 'Venn diagrams' was used, e.g. Comprehensive School Mathematics Program 1978, *CSMP Mathematics for Intermediate Grades. Part III: The Languages of Strings and Arrows.* CEMREL, Inc. (USA). Experimental Version 5 -25401.

[2] See Appendix 1.

[3] Papy (with the collaboration of Frédérique Papy), 1968: *Modern Mathematics, Volume 1*. London: Collier-Macmillan Ltd. (Papy signed his books with his last name only).

Example 2: The metaphor of scales in teaching equations

The metaphor of 'scales' and 'balancing of scales' has been used extensively for the purpose of giving meaning to operations on equations such as, e.g. adding or subtracting the same number to or from both sides of the equation[4]. For the metaphor to work, one must think not of the modern scales, with just one plate, which automatically displays the weight (and sometimes the price) when something is put on the plate, but of the old-fashioned two-plate scales with the product put on one plate and weights on the other. Such scales are, presently, museum objects, and most high school children have never seen them. The drawings of scales in textbooks thus become representations of objects just as abstract as the concept of equation. The understanding of the metaphor requires therefore the teaching of the rules of interpretation of the sequences of drawings of the scales with different objects on both sides and the focusing of the students' attention on whether the plates are drawn on the same or different level - they have no notion of 'balancing the scales'. A teacher might, of course, just ignore the metaphor and teach the operations on equations directly, but some teachers do work with the metaphor for a long time with the students, and produce a whole 'language of scales' to reduce the ambiguity of communication, just as their predecessors were producing the 'language of Venn diagrams'. The risk of ambiguity and unintended interpretations of the scales metaphor is real; there are accounts of this happening in published research papers[5][6]..

Example 3: The use of technology in mathematics teaching: can it favor the 'metacognitive shift' phenomenon?

An important part of the controversy about using or not using calculators and mathematical computer software is concerned with exactly the risk of teaching the teaching aid rather than the mathematics it is supposed to help understanding. In my own practice of teaching linear algebra with Maple, I remember that, at the beginning, I used to spend a lot of time teaching my students the commands of Maple and the quite awkward syntax of the software. Students were spending a lot of time trying to figure out why a command didn't work they way they expected (just to find out, for example, that they had forgotten to put the semi-colon at the end of the command line).

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[4] See Appendix 3: two pages from a Polish tetxbook for Grade 7 students:
Zawadowski, W. et al., 1996: *Matematyka 2001. Podrecznik dla klasy 7*. Warszawa: Wydawnictwa Szkolne i Pedagogiczne. English translation of the text is provided.

[5] Pirie, S.E.B. (1998): Crossing the Gulf between Thought and Symbol: Language as (Slippery) Stepping-Stones. In H. Steinbring, M.G. Bartolini-Bussi, A. Sierpinska (eds.), *Language and Communication in the Mathematics Classroom*. Reston, VA: National Council of Teachers of Mathematics, pp. 7-29.

[6] MacGregor, M. (1998): How Students Interpret Equations: Intuition versus Taught Procedures. In H. Steinbring, M.G. Bartolini-Bussi, A. Sierpinska (eds.), *Language and Communication in the Mathematics Classroom*. Reston, VA: National Council of Teachers of Mathematics, pp. 262-270.

At the end, they were not sure what they are learning in the course: linear algebra or the Maple language. Today, I no longer demand that the students do their homework assignments using Maple; I allow them to do so, if they want. Each student in the class has the right to one-hour tutorial on the use of Maple, and I provide help during my office hours and via e-mail. I am using Maple for teaching: my lectures are written in a Maple worksheet. The worksheet is active during the class; it is projected on a large screen and student generated examples and conjectures are tested using the software, thus avoiding tedious calculations by hand. But, in my lectures, I do not discuss the syntax of the language nor do I make any suggestions of the type, 'Look, here is a useful command'. However, in printing the class notes for the students I do not delete the commands; I was explicitly asked by some students to leave them on, so that they can use them when working with Maple on their own.

It is not clear, however, if technology can be classified as a 'teaching aid', aimed at overcoming the difficulties in the weak student and enhance understanding in the stronger student. In the mathematician's hands technology is an instrument, useful or sometimes even indispensable in the execution of certain tasks. And then, the teaching of how to use technology in doing mathematics is not a symptom of meta-cognitive shift. It is the teaching of the craft of using a tool in an intelligent way. Some research which has already been done on the teaching and learning of mathematics with technology[7], shows that for many students the calculator is the ultimate reference and not a 'mathematical instrument' in solving problems. This research also points to the need of preparing the students to correctly interpret and use computer's outputs.

For example, if I ask Maple to solve the equation $x^2 - (1+\sqrt{2})x + \sqrt{2} = 0$, I obtain 1, and $\sqrt{2}$. But when I ask Maple to verify if $\sqrt{2}$ satisfies the equation, the response is 'false' (see the bordered figure below). One has to understand that for Maple, $\sqrt{2}$ does not exist; '$\sqrt{2}$' is only a symbol for a decimal number, an approximation of $\sqrt{2}$. But no approximation of $\sqrt{2}$ satisfies the above equation.

[7] see, for example, Guin, D. & Trouche, L. (1999): The Complex Process of Converting Tools into Mathematical Instruments: the Case of Calculators. *International Journal of Computers in Mathematical Learning 3*, 195-227.

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> f:=x->x^2-(1+sqrt(2))*x+sqrt(2);
                               2
                f := x -> x  - (1 + sqrt(2)) x + sqrt(2)

> solve(f(x)=0,x);
                                  1, √2
> evalb(f(1)=0);
                                   true
> evalb(f(sqrt(2))=0);
                                  false
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If I ask the calculator TI-92 to solve the same equation in the exact mode, the response I obtain is '$x = -(\sqrt{3-2\sqrt{2}} - \sqrt{2} - 1)/2$ or $x = (\sqrt{3-2\sqrt{2}} + \sqrt{2} + 1)/2$'. Of course, $3 - 2\sqrt{2} = (1 - \sqrt{2})^2$ and the same numbers 1 and $\sqrt{2}$ are obtained, but the calculator will not spontaneously simplify an expression. Besides, one has to know that the calculator is simplifying an expression upon the command 'expand' and it's no use looking for a 'simplify' command, because it does not exist.

2. THE META-MATHEMATICAL SHIFT

This phenomenon, only briefly mentioned by Brousseau (p. 39), consists in the 'replacing [by the teacher] of a mathematical problem by a discussion of the logic of its solution and attributing all sources of error [in its solution] to [a misunderstanding of this logic]'. An example of this could be the situation where the teacher, in the aim of improving the students' performance in solving equations and understanding what they are doing, teaches the students a theory of equations: gives a definition of an equation, and before that, a definition of a variable, an algebraic expression, the logical axioms of equality (reflexivity, symmetry and transitivity) and the algebraic properties of equality in number systems (e.g. if $a = b$ and $c = d$ then $a + c = b + d$), etc. Such knowledge belongs to the so-called 'meta-mathematics', i.e. a theory of the language of mathematics, which makes abstraction from the intuitive meanings of particular mathematical statements or objects and occupies itself only with their general form.

The New Math reforms were fraught with proposals amounting to no more and no less than teaching meta-mathematics to school-children; after all, logic and theory of relations can be regarded as parts of meta-mathematics. Today, the shift is less felt, but some elements of it still exist, I believe, in the introduction of equations in secondary schools via the notions of 'open' and 'closed sentences'.

3. The implicit suggestion of analogy

Most of the phenomena of teaching mathematics we have reviewed so far represent the different ways in which teachers try to

(a) maintain the fiction that learning does, indeed, take place, and that, therefore, they are doing what is expected of them as teachers (Topaze, Jourdain), or,

(b) genuinely help the students learn better, but the method chosen does not and cannot bring about the expected results (Dienes, the meta-cognitive and the meta-mathematical shifts).

The phenomenon discussed in this section (p. 27) belongs to the category (a) of these phenomena. The teacher gives the students problems formulated so as to highlight their analogy with problems previously solved by the teacher on the board or discussed in class. The teacher does not want to explicitly say, 'solve this problem just like we solved problem number so and so' but she gives a hint, sometimes just by asking the question in exactly the same form. The students are expected, in this game, to get the hint. The teacher feels miserable when they don't because this forces him or her to explicitly point to the analogy and the falsehood of her game is revealed.

4. The paradox of learning by adaptation

The remarks that Brousseau is making in the section 'Paradoxes of learning by adaptation' (pp. 44-47) could be understood as a criticism of the constructivist epistemology and psychology of learning and a promotion of the interactionist stance in these matters.

Constructivism as a psychology and epistemology made its way into mathematics education in the late 70s and, in the USA, it generated a lot of basic research into children's processes of acquiring a notion of natural number, fractions, arithmetic operations, and also more mature students' processes of learning such notions as the exponential function[8]. It developed into a certain 'ideal' of teaching mathematics, where there would be no lecturing, no drill exercises, but individual children 'constructing their own knowledge' (a well known constructivist slogan) by solving problems, with the teacher's role reduced to that of an interviewer ('Tell me how you solved this problem'). By *adapting* to a stimulating environment, with interesting problems, children's 'cognitive structures' would grow and evolve in a natural

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[8] See, for example, the volume:
Steffe, L.P. & Gale, J. (Eds.), 1995: *Constructivism in Education*. Hillsdale, New Jersey: Lawrence Erlbaum Associates, Publishers.

way. The metaphor here is that of a biological organism changing through adaptation to its environment.

> From the constructivist perspective, as Piaget stressed, knowing is an adaptive activity. This means that one should think of knowledge as a kind of compendium of concepts and actions that one has found to be successful, given the purposes one had in mind. This notion is analogous to the notion of adaptation in evolutionary biology, expanded to include, beyond the goal of survival, the goal of a coherent conceptual organization of the world as we experience it. (Von Glasersfeld, 1995, p. 7)[9].

Constructivism has developed in opposition to Behaviorism, a theory of learning quite prominent in the USA, based on the fundamental assumption that a rewarded response is the action that will be repeated, 'reinforced'. Behaviorism led to a theory of instruction based on drill and practice and a system of rewards and punishments.

Constructivism attracted mathematics educators interested not only in early childhood education, but also those working with university students. For example, the so-called APOS[10] theory of learning mathematics advanced by Ed Dubinsky and his collaborators, is firmly based in the constructivist epistemology. Here is how Dubinsky defines mathematical knowledge:

> Mathematical knowledge is an individual's tendency to respond, in a social context, to a perceived problem situation by constructing, re-constructing and organizing, in her or his mind, mathematical actions, processes, objects and schemas with which to deal with the situation (Dubinsky, 1997, p. 95)[11].

Thus, for a constructivist, knowledge is a psychological entity: an individual's network of cognitive structures, schemas, constructed through the individual's experience in solving all kinds of problems, practical and theoretical. From this perspective, objectivist notions such as 'truth' and 'validity' of knowledge which refer to a 'correct representation of reality', do not make sense; they are replaced, in constructivism, by the notion of 'viability'. The very concepts of 'correct' and 'reality' are questioned by constructivism.

> To the biologist, a living organism is viable as long as it manages to survive in its environment. To the constructivist, concepts, models, theories, and so on are viable if they prove adequate in the contexts in which they were created. Viability - quite unlike truth - is relative to a context of goals and purposes. But these goals and purposes are not limited to the concrete or material. In science, for instance, there is, beyond the goal of solving specific problems, the goal of constructing as coherent a model as possible of the experiential world (Von Glasersfeld, ibid.).

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[9] von Glasersfeld, E. (1995): A Constructivist Approach to Teaching. In L.P. Steffe & J. Gale (eds.), *Constructivism in Education*. Hillsdale, New Jersey: Lawrence Erlbaum Associates, Publishers, pp. 3-15.
[10] 'APOS' stands for Action - Process - Object - Schema.
[11] Dubinsky, E. (1997): Some Thoughts on a First Course in Linear Algebra at the College Level. In D. Carlson, C.R. Johnson, D.C. Lay, A. Duane Porter, A. Watkins, W. Watkins (eds.), *Resources for Teaching Linear Algebra*. The Mathematical Association of America, MAA Notes, Volume 42, pp. 85-105.

Brousseau holds a very different view of learning and knowledge. He claims that the notion of learning by adaptation is inconsistent (p. 44-45). '*Adaptation* - he claims - *contradicts the idea of new knowledge'*. If a person solves a problem different from all the problems she had solved so far, with some adaptation of the knowledge she already had, why would she think she has invented some new knowledge? In a similar situation she could go, if necessary, through the same process, from scratch, and therefore there is no need to identify the process as a new 'method' or 'new knowledge'. But if this person shows her way of solving the problem to some other persons who were also trying to solve it but were no able to, then their interest in it, eagerness to understand it, and their appraisal of its more general and not just local value, will indicate to her that some new knowledge has actually been invented. This is what appears to be meant by the statement: '*knowledge is almost the cultural recognition that direct knowing is impotent to solve some situations naturally (by adaptation)*' (p. 45).

This view of knowledge is strongly reminiscent of the position taken by interactionism in social psychology. Interactionism in social psychology has its roots in the pragmatic positions of Peirce, James, Dewey (USA, end of the 19th and beginning of 20th century) and the sociological research of the so-called Chicago school of sociology[12]. Its epistemology stresses the roles of experience, the common sense, and the existence of multiple interpretations. All one knows is experience; but this experience is not a sequence of isolated sensations but a culturally shared world which we take for granted in the everyday life. Experience is thus the common sense, where the 'common' means the 'shared'. And 'shared' means 'objective'. We do not want to have 'private knowledge' or 'subjective knowledge' - we do not value it. When we notice that our knowledge is different from the shared knowledge, because, for example, we are unable to achieve goals that others achieve, we treat this knowledge as something subjective and we reconstruct it until it allows us to achieve the goals. Then we treat it as objective again (Hammersley, 1989, p. 56). The basic unit of the interactionist psychology is a goal directed action; it is assumed that our actions are social, even if we perform them alone, because we are able to view ourselves as objects. In undertaking an action we imagine the effects that it will produce on others and how they could react (ibid., p. 59):

> It is this socially generated ability to view oneself as an object, and to interpret the world in alternative ways, that allows people to modify their interpretations and to choose different courses of action. As a result, human action… is constructed through a reflexive process which takes the form of a person making

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[12] See an interesting account of interactionism in
Hammersley, M. 1989: *The Dilemma of Quantitative Method. Herbert Blumer and the Chicago Tradition*. London: Routledge.

indications to himself, that is to say, noting things and determining their significance for his line of action. (Blumer, cited in Hammersley, ibid., p. 130).

Applying this theory to learning, one might perhaps say that, from the interactionist perspective, new knowledge coincides with choosing different courses of action, and different interpretations of a class of situations, and this can happen only when the learner sees himself or herself as an object - a member of a society and judges his or her actions from that external point of view. In a sense, one can learn new knowledge if one is, at once, a learner, and one's own teacher. In a sense, also, one knows only if one 'acts' one who knows - 'if one acts as a knowledgeable person', according to what the society takes for granted as the behavior of a knowledgeable person. Only 'shared' or 'public', or culturally identified as such knowledge is considered to be *knowledge*.

5. THE PARADOX OF THE TEACHER AS LEARNER

For constructivists, the ideal teaching situation is one where the teacher does not know the solution of the problem given in a didactic situation. The problem could have been invented by the student, as a result of being in some more general problem situation, and the teacher and the student work together on it as partners-in-mathematics.

Brousseau compares this situation to one in a theater, where the actor would not just act a feeling (e.g. joy or anger) but actually experience this feeling. Referring to Diderot's analysis of acting, Brousseau claims that this would result in quite poor performance: such acting might not be very convincing: '*the more the actor feels emotions he wants to play, the less he is able to allow the audience to share this feeling'* (p. 46). Being on stage, visible for the audience, the actor could become ashamed of his private feelings and try to conceal their perceptible symptoms rather than amplify them, which is what he has to do if he wants the audience to understand what is going on.

The point Brousseau is trying to make, I presume, is that if a teacher finds it useful to act as if she did not know how to solve the problem, this should only be good acting and not the actual state of the teacher's mind. Not only should the teacher know the knowledge that she intends to teach but she should use all the means in her repertoire of 'didactic tricks' to put on stage and *devolve* (p. 31) to the students a problem situation they would consider it their own responsibility to solve and which would lead them to develop or use the knowledge in question.