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THEORY IS NOT NECESSARY.
PRACTICE OF THEORY IS.

On the necessity of practical understanding of theory

A shorter version of this piece was presented at the ICME-10 Topic Study Group 22, "Learning and cognition in mathematics: Students' formation of mathematical conceptions, notions, strategies, and beliefs", as a contribution to the sub-theme, "Models of mathematical thinking and understanding".

My aim in the talk was to bring up, for discussion, the following question:

What is the use,
for didactics of mathematics,
of ***general*** models
of mathematical thinking and understanding?

What is the use,
in particular,
of the classic epistemological distinction
between *practical* and *theoretical* thinking?

This distinction, as I will try to show in this paper, is a very crude model of an extremely complex reality. More refined models are needed for the purposes of didactics, and even these could be uninformative unless complemented by epistemological analyses of the particular mathematical subject matter.

Examples of students' work in linear algebra will show that successful students of university level mathematics certainly have a sense of what it means to work within a theoretical system. But they are also very "practical" in moving about the theory, capitalizing on previous experience without trying to simply reproduce a method learned by heart, and picking exactly what is needed from the theory to get straight to the solution. They are ready to give up on rigor, consistency of notation, generality of the solution, if this is not absolutely necessary for obtaining a satisfactory solution. They solve the problem at hand; they do not develop a theory of solving all problems of a kind, yet there is an undercurrent of generalizable techniques in their solutions. Perhaps this could be classified as a kind of "situated knowledge". This is why analyzing their solutions requires getting into the mathematical contents of the problem. And it is the content specificity of these analyses that, I think, makes them useful for didactics of mathematics.

THEORY AND ART OF MATHEMATICS

Aristotle (*Metaphysics*, Book A, 981b) characterized theoretical knowledge by, among others, the possibility of its teaching - presumably, in a direct, verbal way. But the art of inventing and using this knowledge cannot be taught this way. And this is the source of all our didactic problems in mathematics education.

Teaching the theory of mathematics is not a problem.
Any university professor can do that.
The problem is
how to teach the art of mathematics.

The art of doing mathematics is a complex thing and there is little systematic, yet content specific, knowledge about it. There are many examples, however, to which mathematicians and mathematics educators refer using a variety of names.

Mathematicians speak about "techniques" and "methods". In sketching an outline of the mathematical techniques that pervaded the 20th century mathematical developments Atiyah (2000/2003) mentioned, among others, systematic construction of linear (or abelian) invariants of non-linear (or non-abelian) situations; for example, studying polynomials (non-linear objects) from the point of view of vector spaces (linear objects) that they can generate. These techniques were systematized as theories (in particular, homology theory and K-theory). As postulated in Chevallard's theory of scientific practice (Chevallard, 1999), techniques of solving particular problems become quickly overgrown with theories of increasing generality.

In the context of mathematics education, the art of mathematics refers to thinking involved in solving more or less conventional or imaginative, closed or open, formal or informal exercises and problems.

For example, Mamona-Downs & Downs (2004) speak about "specific problem solving techniques" (as opposed to general heuristic principles such as those proposed by Polya) and raise the issue of the possibility of devising "pedagogical approaches that should help students extract a technique from previously met paradigms". They have designed and tried such approach for teaching the technique of using bijections between finite sets to solve certain combinatorial problems: to calculate the number of elements in a complicated set, it may be useful to map it in a one-to-one fashion onto a simpler set, whose number of elements can be calculated quite easily. The experiment showed that the teaching of mathematical techniques is much more difficult than anticipated.

Castela (2004) refers to something similar using the expression "knowledge about mathematical economy" (*connaissances sur le fonctionnement mathématique*, in French), which she defines as "knowledge about the different roles played by mathematical objects in the solution of already solved problems". This knowledge about mathematical economy enable students to recognize a problem as similar to one they have already solved and use a similar method or "trick" to quickly and efficiently solve the problem at hand. This knowledge, again, is not assumed to include meta-cognitive skills and strategies (such as those postulated by Polya and Schoenfeld), but is confined to mathematical ideas and techniques that appear in the final solution of the problem. Castela's examples of knowledge of mathematical economy include: "to prove that three lines are concurrent, you may take the intersection point of two of them and prove that it belongs to the third one"; "to prove that a determinant of a matrix is zero, you may expand it or prove that the set of its columns (or rows) is linearly independent".

Theoretical and practical thinking distinction – Act I

In my and my colleagues' research, the main distinction was not between theory and art, or theoretical and practical *knowledge*, but between theoretical and practical *thinking.* We were more interested in the act than in the product of knowing (see Sierpinska et al., 2002, Chapter 1). This distinction appeared in our discussions about the reasons of a very limited success of our teaching experiments aimed at helping students understand linear algebra and especially vector space theory and linear transformations (Sierpinska, Dreyfus & Hillel, 1999; Sierpinska, 2000). Students, with whom we worked, were inclined to interpret our dynamic geometrical representations of vectors and linear transformations in a literal (iconic) rather than symbolic way. They had trouble using definitions of concepts in reasoning and preferred thinking about them as families of examples (e.g. a linear transformation is a rotation, a dilation, or a combination of these).

These and other aspects of their thinking appeared as violations of what has traditionally been characterized as features of theoretical thinking (for other examples of non-theoretical thinking, see, e.g. Brown, 1981, see dialogue pp. 28-29; Tall & Chin, 2002). Our research, mentioned above, involved students who were not, in general, very strong in mathematics in terms of their previous academic achievement.

But then we focused on identifying high achievers' tendency to thinking theoretically (according to a definition of theoretical thinking, derived from some classical understandings of the notion, Sierpinska et al., 2002, Chapter 1): we interviewed 14 students who achieved A grades in a first university linear algebra course (vector spaces, linear transformations, eigenvalue theory and diagonalization). It turned out that, on the average, these students' tendency to theoretical thinking was not very high. It appeared that theoretical thinking is not all that necessary for high achievement. On the other hand, we could prove that, according to our definition, theoretical thinking is necessary for understanding linear algebra. Our conclusion was that there must be something wrong with the courses and assessment, and we analyzed the final examination questions. Indeed, their solution did not require much theoretical thinking.

THEORETICAL AND PRACTICAL THINKING DISTINCTION – ACT II

As an instructor in these first year linear algebra courses I decided to foster theoretical thinking in students by administering weekly quizzes, each with two short conceptual questions, which seemed to require quite a bit of this kind of thinking. These courses are attended mainly by students who are selected for their high grades in mathematics in their pre-university education, and having previously taken an elementary "matrices and vectors" course.

Already the first quizzes revealed that these students massively refer to definitions rather than to paradigmatic examples, and they spontaneously justify their solutions (using mathematical argumentation), even if not explicitly asked to. They know that a few examples do not count as a proof of a general statement. They all study the theory, and try to understand and write proofs.

Yet, not all were getting their solutions right. A closer analysis of these students' solutions suggested to us that...

...those who obtained correct solutions
were not only "theoretical thinkers";

they also had developed
a "practical understanding of theory".

There will be examples of this type of thinking later in this paper. But first I will shortly describe the model of theoretical thinking we have used in our research, so that it is clear what I have in mind when I claim that theoretical thinking is used in this or that solution.

A MODEL OF THEORETICAL AND PRACTICAL THINKING

In our research, mathematical thinking was assumed to be based on an interaction of practical and theoretical thinking. Practical thinking was considered to be the source of wonder and curiosity leading to bold conjectures, which then provided food for theoretical thought. Our definition of theoretical thinking was, therefore, based on a number of oppositions related to

- reasons for thinking;
- objects of thinking;
- means of thought;
- main concerns;
- products of thinking,

which distinguished it from practical thinking (see Table 1).

TABLE 1.

DISTINCTION BETWEEN PRACTICAL AND THEORETICAL THINKING

Practical thinking	Theoretical thinking
Reasons for thinking (Why am I thinking?)	
To get things done. To understand something. To describe reality. To solve the problem at hand.	To put some order in one's thinking. To systematize one's understanding of something. To construct a consistent model of reality. To devise a method of solving all problems of a certain type.
Objects of thinking (What am I thinking about?)	
Particular things, matters, events, people.	Systems of concepts.
Means of thought (What representations do I use to think?)	
Ad hoc representations. Visual imagery. Geometric configurations. Calculations of concrete results using known and unquestioned techniques.	Systems of representation. Linguistic systems and symbolic notations. Consistent geometric models. Devising better adapted or more general techniques for solving a range of problems.
Concerns (What is important?)	
Significance of actions and facts. Factual and social connections: contingency in time and space, analogy between circumstances across time, particular examples, and personal experience. Factual and social validity: the proof of a plan of action is in the outcomes of the action. Realistic reasoning: Taking into account only the most natural and plausible cases. Technical concerns: making sure an alternative course of action is available, of the chosen one doesn't work.	Meanings of concepts. Conceptual connections: consequences of assumed meanings for the meanings of other concepts. Epistemological validity: internal consistency of systems of symbolic representations. Hypothetical reasoning: Taking into account all logically possible cases. Methodological concerns: to have a theory of the possible and logically legitimate courses of action (a "methodology"
Outcomes of thinking	
Statements of fact. Decisions about what to do next.	Hypothetical (conditional) statements. Theories. Specialized notations.

As one can imagine from the above description, theoretical thinking is not very "practical" in the common sense of the word: theoretical thinking, as such, is not more economical in terms of time and more effective in terms of the success of actions that would be undertaken on its basis. A strictly theoretical thinker would be a pathology: he or she would never be able to make a decision, endlessly considering the different logically equivalent possibilities of action.

Therefore, the distinction between pratical and theoretical thinking can be only methodological. Reasoning based on purely conceptual connections and epistemological validation does not exist in the reality of scientific cultures. Factual and social significance, and the epistemological meanings and validations are difficult to separate because the latter enjoy existence only to the extent that this existence is socially recognized (through publications, for example). Some meanings and reasonings are chosen over other, not only because they are more logical or consistent with the rest of the system, but because they appear – to the practitioners of the theory – as more "economical", more "elegant", and more "interesting", which are socio-cultural reasons.

In (Sierpinska et al., 2002, Chapter 1), we were saying:

> Thus, understanding mathematical theories requires both practical thinking and theoretical thinking. Practical thinking is the ground against which theoretical thinking acquires its reason of being and without which it loses its epistemological significance. It is in this sense that practical thinking is an *epistemological* obstacle. It is *a cognitive and cultural phenomenon, which stands in the way of certain developments in mathematics, while being, at the same time, an indispensable component of the construction of mathematical knowledge* (Sierpinska, 1990; 1992; 1994).

But my work with classes of generally mathematically strong and motivated students made me aware of a duality in this "epistemological obstacle" relationship: theoretical thinking can also function as an obstacle to practical thinking in mathematics. We will presently see a few examples of this phenomenon.

OF THE IMPORTANCE OF PRACTICAL UNDERSTANDING OF THEORY

The examples of students' solutions discussed in this section are meant to illustrate the claim that theoretical thinking is necessary, but not sufficient for high achievement in university level linear algebra courses.

One needs a lot of visual imagery and technical skills to be an efficient problem solver.

All examples are based on students' solutions to 15-20 minutes quiz questions in a second undergraduate linear algebra course. The course focused on two main ideas: Inner Product Spaces and canonical forms of linear operators.

Example 1. Proving that an assumption in a theorem is essential

This example is taken from the third quiz on Inner Product Spaces; question 1 of 2 (students had 7-10 minutes to solve it). The question belonged to the following type of questions: How essential is a given assumption in a given theorem, known by the students from the lecture.

The question

Given the statement:

Let V be an Inner Product Space over R, and <.,.> the inner product on V.

Let $B = \{u_1, ..., u_n\}$ be an orthogonal basis of V.

Then any vector v in V can be represented as $v = \sum_{i=1}^{n} \frac{\langle v , u_i \rangle}{\langle u_i , u_i \rangle} u_i$

Would this statement remain true if B was not assumed to be orthogonal? Justify your answer.

Discussion of the possible reasonings about the question

Here is a possible solution to this question.

Not true.

Take, for example, $V = R^2$ with dot product, and the basis B= $\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \}$ which is not orthogonal. Take $v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Then $\sum_{i=1}^{n} \frac{\langle v , u_i \rangle}{\langle u_i , u_i \rangle} u_i = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 1.5 \\ 1.5 \end{bmatrix} = \begin{bmatrix} 3.5 \\ 1.5 \end{bmatrix}$

But this vector is different from v, so the equality does not hold in general.

How can one come up with this solution?

A priori, there are (at least) the following two ways of thinking about this problem, depending on how one comes to guess that the assumption of orthogonality is essential and therefore decides to look for a counter-example rather than try to prove the statement without the assumption of orthogonality.

Reasoning 1 - Visual

The guess comes from visualizing the theorem as a generalization of representing two-dimensional arrow-vectors in a coordinate system as a sum of their projections on the axes: the terms of the sum in the theorem are nothing but projections of v on the directions of each u_i. If the axes are not orthogonal, then the projections do not add up to the vector (Figure 1). This visualization suggests a simple counter-example in R^2 with dot product, which is a "typical" inner product space; the choice of non-orthogonal basis is guided by visualizing a pair of oblique vectors.

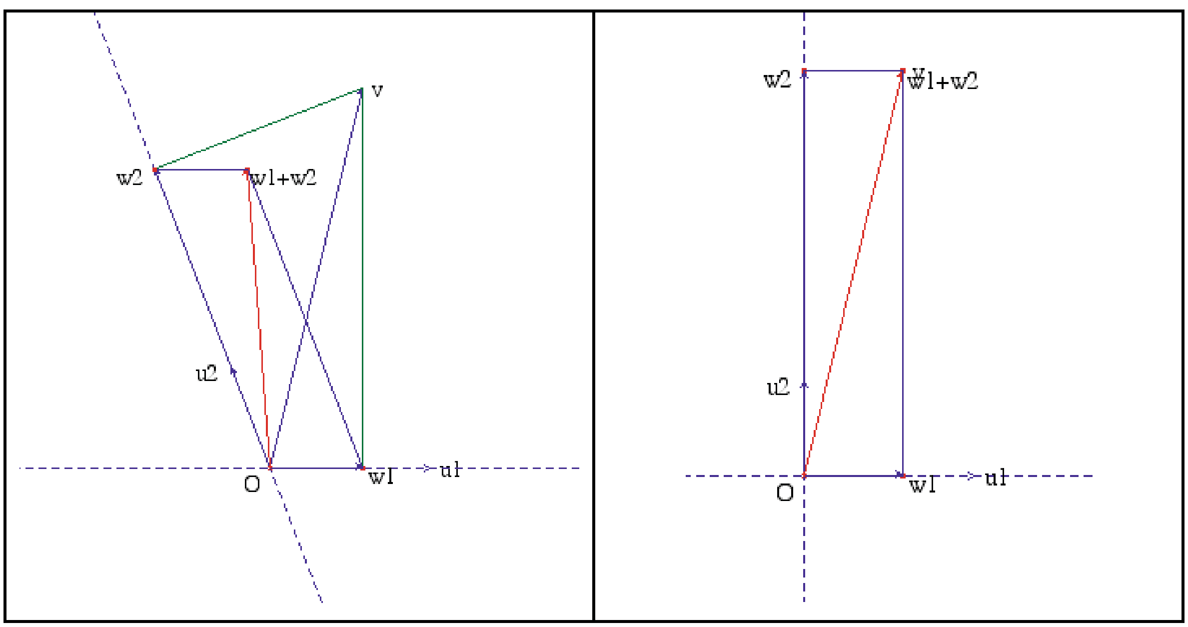

Figure 1. If u_1 and u_2 are not orthogonal (perpendicular) then v is not the sum of its projections on u_1 and u_2: For a line, which is perpendicular to u_2 to be, at the same time, parallel to u_1, u_1 and u_2 must be orthogonal.

Comment about Reasoning 1. It can be argued that thinking involved in this solution has quite a few characteristics of practical thinking. First of all, it is aimed at a quick solution of the problem at hand; it has to be quickly (7-10 minutes!) decided whether the modified statement would be true or false, and why. The solver does not engage in generalizing the problem and doing more than finding a correct answer. Instead of trying to produce and analyze proofs of the given statement and of its stronger version (without the assumption of orthogonality of B) to decide about the logical necessity or not of the assumption, the solver tries to guess the answer, based on visualizing the situation in a simple and well-known particular case. In general, visualization is much faster than the production of linguistic forms such as proofs. But also, a visual understanding of the theorem, as a generalization of a simple geometric fact about orthogonal and parallel projections, is more useful than the general axiomatic definitions of IPS and orthogonal bases, and it immediately suggests that the answer is negative. The solver chooses to think about R^2 with dot product because, in this context, theory and geometric intution agree to a certain extent. Theorems feel like "facts" one is well familiar with, and it is not necessary to refer to formal definitions to derive conclusions. In particular, one doesn't have to use a formal definition of linear independence to find a basis for R^2, and one doesn't have to calculate anything or use the definition of orthogonality to make sure this basis is not orthogonal. Calculating dot products is easy and the notation is familiar, so it is not necessary to pay much attention to symbolic conventions.

On the other hand, the solver must have a certain logical sensitivity to be satisfied with this solution: to negate a general statement, it is, indeed, enough, to produce a single example of an object which satisfies the premises but not the conclusions. It is not necessary to prove, for example, that, if B is not orthogonal, v is "never" equal to the sum of its projections on the vectors of B.

Reasoning 2 – Formal

This reasoning is based on an analysis of the proof of the statement. Where is the assumption of orthogonality of B used in the proof of the statement? Since B is a basis, any vector v has a representation

$$v = a_1 u_1 + ... + a_n u_n \ (*).$$

Up till now, we haven't even used the fact that V is endowed with an inner product. But then we go on to show that the coefficients a_i are equal to

$$<v, u_i>/<u_i , u_i>.$$

This means that $a_i <u_i , u_i> = <v, u_i>$, which suggests the representation of $<v, u_i>$ in two ways, using the equation (*):

$$<v, u_i > = < a_1 u_1 + ... + a_n u_n, u_i >$$

Using the linearity property of the inner product, we get

$$<v, u_i > = a_1 < u_1, u_i > + ... + a_n <u_n, u_i >.$$

If B is orthogonal, all terms but $a_i <u_i , u_i >$ on the right hand side of the equation vanish (by definition of orthogonality) and the proof is done. If B is not orthogonal, then the relationship cannot be thus simplified and, even if the modified statement is true, it cannot be proved this way. There is a possibility, therefore, that the modified statement is not true, and it makes sense to look for a counter-example.

Comment about Reasoning 2. This reasoning has several characteristics of theoretical thinking. It is discursive, not visual, thinking. It is a reasoning about a reasoning (an analysis of a proof). There is reference to the definitional properties of the inner product, and of orthogonality. There is processing of expressions in a coherent notational system.

But theoretical thinking alone would be too cumbersome to produce the counter-example. A choice must be made from among the hypothetical infinity of possible examples, which are all equally valid. It is experience, familiarity with some of them, that makes the choice possible, but this experience and familiarity are features of practical, not theoretical thinking.

Therefore, whatever the route one takes to start looking for a counter-example, finding the counter-example requires a good dose of practical thinking.

Students' solutions

Quite a few students in my class produced counter-examples as in the cited solution, using R^2 with the dot product. These students did not write how they arrived at looking for the counter-example, and one can only guess that some of them used the visual and some – the formal approach. But there were also quite a few students, who just stated that the modified statement would be false, because the proof, which they had seen in class, would not work, and they did not produce a counter-example. For example, one of the students wrote,

> If B is not an orthogonal basis of V
> then the statement is not true
> because if we take the inner product,
> the remaining items $\neq 0$
> $\langle u_1 + \dots + u_n , v \rangle = \langle u_1 , v \rangle + \dots + \langle u_n , v \rangle$

Thinking underlying the above cited solution may have some features of theoretical thinking, such as being discursive rather than visual, and being a reasoning about a reasoning; there is also a reference to the definition of orthogonality.

But there is no awareness of the hypothetical possibility of an equally valid alternative proof of the statement, where the assumption of orthogonality would not be used. There is an implicit assumption that the proof given by the teacher is the only possible proof.

Of course, there may be some practical thinking underlying the solution: even if one doesn't have enough experience with examples of inner products, this theoretical approach quickly suggests and justifies, if not proves, a correct answer, which – the student can argue - merits at least 50% of the marks for the question.

There were also other solutions to this question. For example, this one:

> No, this representation $v = \sum_{i=1}^{n} \frac{\langle v \ , \ u_i \rangle}{\langle u_i \ , \ u_i \rangle} \ u_i$
> is linearly independent,
> and when the basis is assumed to be orthogonal,
> it is necessarily linearly independent to any vector $v \in V$.
> If it is not assumed,
> the basis can be independent or dependent,
> and if dependent, then $v \in V$ can't be represented as above.

The thinking underlying this solution also has many features of theoretical thinking (discursive, based on conceptual connections, concerned with epistemological validity, with a systemic approach to symbolic representations). Unfortunately, the conceptual connections are, most of them, wrong, as can be inferred from the incorrect "grammar" of the discourse about linear independence and basis. In this case, the student is not lacking in predisposition to theoretical thinking. But she is lacking in understanding the particular concepts of the particular theory. Maybe she needs to engage with examples and illustrations of the theory some more to get her conceptual understanding straight.

Example 2. True or false question about an equality

Second quiz, question 2, about Inner Product Spaces.

The question

In the statement below, u, w are vectors in an IPS; the norm is defined by

||v|| = sqrt(<v,v>).

STATEMENT:
If the set {u, w} is linearly dependent then ||u + w|| = ||u|| + ||w||.

Is the statement true or false? Justify.

Discussion of possible reasonings about the question

Again, one can approach the problem visually or formally. However, in this case, the visual approach must be supported by an element of theoretical thinking (considering all logically possible cases and not only one most "natural" case) to produce the correct answer.

Solution 1 – Visual

To get a geometric sense of the statement, we visualize the situation in R^2 with dot product, represented in a Cartesian plane. We visualize vectors as arrows and their norm – as the length of the arrow. In this case, two vectors are linearly dependent if the arrows, located at the origin, are collinear. The sum of two vectors is obtained by putting the arrows head to tail. It is natural to visualize the two vectors as oriented in the same way. In this case, the length of the sum is, obviously, equal to the sum of the lengths of the component vectors.

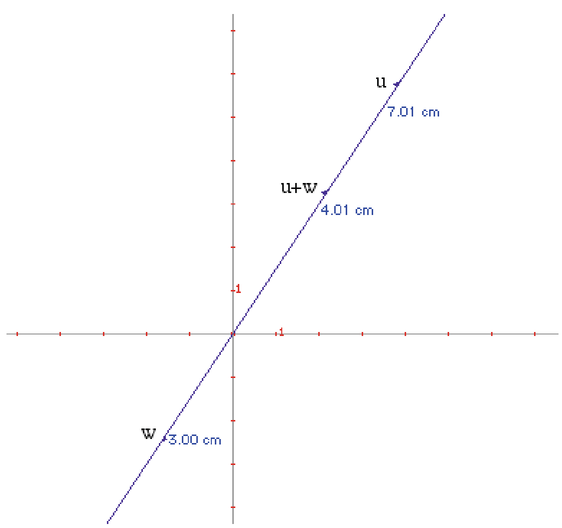

Figure 2. The norm of the sum of two vectors is not always equal to the sum of their norms

If this is the only case one can think of, the answer is that the statement is true, which is incorrect. One needs to think of all logically possible mutual positions of the two linearly independent vectors, not only of the "most natural" position. This leads to considering the case of the vectors being oriented in opposite ways and seeing that the statement is not true in general (see Figure 2).

This "theoretical visualization" suggests looking for a counter-example with two oppositely oriented vectors, which can then be given in the algebraic language of R^2 and dot product.

I have no proof that this is how students who gave solutions such as the one below used this kind of thinking, because they simply just gave a counter-example in R^2 with dot product but did not explain how they arrived at this solution.

$$u = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad w = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

$$||u + w|| = \sqrt{\begin{bmatrix} 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix}} = \sqrt{2}$$

$$||u|| + ||w|| = \sqrt{\begin{bmatrix} -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}} + \sqrt{\begin{bmatrix} 2 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -2 \end{bmatrix}} = \sqrt{2} + \sqrt{8}$$

$$\sqrt{2} \neq \sqrt{2} + \sqrt{8}$$

$\therefore$ **False**

Solution 2. Formal

This solution is based on deriving conclusions from the assumption of linear dependence and properties of operations in vectors spaces and of inner products in general. Linear dependence of u and w yields the existence of a real number k such that, say, u = kw.

Then $||u + w|| = ||kw + w|| = ||(k+1)w|| = |k+1|\,||w||$. On the other hand, $||u|| + ||w|| = (|k| + 1)||w||$.

Since $|k+1|$ is not always equal to $|k| + 1$, the statement is not true in general. There is no need to give a counter-example for $|k+1|=|k|+1$, because this can be assumed to be a well-known elementary result.

Many students produced solutions based on this kind of theoretical reasoning in general terms. But quite a few of them made the mistake of thinking that $||kw||= k\ ||w||$, forgetting the absolute value of k. They concluded that the statement is true.

There were also other mistakes in the students' solutions. Here is an example:

If the set {u, w} is linearly dependent ⇒ u = aw where a ∈ R.

$$||aw + w|| = ||(a+1)w|| = (a+1)||w|| = (a+1)\sqrt{\langle w\ ,\ w\rangle} = (a+1)\sqrt{w_1^2 + \ldots + w_n^2}$$

$$||aw|| + ||w|| = a||w|| + 1||w|| = (a+1)||w|| = (a+1)\ \sqrt{w_1^2 + \ldots + w_n^2}$$

TRUE

The generality of this reasoning is not complete; the inner product is understood as the dot product. It also shows gaps in knowledge of basic theoretical facts. The approach is theoretical to a certain extent, but insufficient theoretical knowledge prevents the student to obtain a correct answer.

There were also students who used the formal approach and arrived at the correct answer, but their reasoning was flawed. For example, in the following solution, the student did not forget about the absolute value in $||kw|| = |k|||w||$, but she appeared to believe that absolute value is an additive function and, when squaring a product, she squared one term but not the other. However, she appeared to know that square root is not an additive function and used this fact to conclude that the statement is false.

If u and w are linearly dependent then w = ku for k ∈ R.

$$||u + w||^2 = ||u + ku||^2 = |1 + k|\ ||u||^2 = ||u||^2 + |k|||u||^2$$

$$\therefore ||u + w|| = \sqrt{||u||^2 + |k|\ ||u||^2} \neq ||u|| + ||w||$$

False.

It is not enough to have a tendency to theoretical thinking to achieve well in linear algebra. One also needs knowledge of basic theoretical facts and a good range of technical algebraic skills.

The necessity of interaction between theoretical thinking, technical thinking and theoretical knowledge in solving linear algebra problems is well seen in the following example.

Example 3.

Second quiz, question 1, about the minimal polynomial of a linear operator (7th quiz from the beginning of term).

The question

Let $T: R^2 \rightarrow R^2$ be a linear operator such that $T(\begin{bmatrix}1\\2\end{bmatrix}) = \begin{bmatrix}-1\\0\end{bmatrix}$ and $T(\begin{bmatrix}3\\4\end{bmatrix}) = \begin{bmatrix}1\\8\end{bmatrix}$. What is the minimal polynomial of T?

Discussion of the possible reasonings about the question

To solve this problem quickly (7-10 minutes), it is not enough to know the definition of linear operator and of the minimal polynomial. One needs to know the consequences of these definitions and one needs to be familiar with the technique of representing linear operators by matrices relative to a basis, and calculating the minimal polynomial from the characteristic polynomial. In this case, it is useful to know that if the characteristic polynomial is irreducible over R, then the minimal polynomial is equal to the characteristic polynomial (modulo the sign, if the characteristic polynomial is not monic). It is also useful to recognize a basis in R^2 when seeing one, without any calculations. And it is useful to guess how a few concrete vectors can be represented as linear combinations of two given vectors, without solving the general problem of finding the coordinate vector of an arbitrary vector relative to a basis. One needs to be so familiar with the theory, as to use it as part of one's practical knowledge.

In solving the problem, one can represent T in the basis $B = \{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}3\\4\end{bmatrix}\}$ (it is seen without any calculation or proof that this is a basis, for someone with some experience in linear algebra: there are two vectors and neither is a multiple of the other), or in the standard basis $S = \{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\}$. In principle, one should know that it doesn't matter which basis is chosen, because the minimal polynomial is invariant with respect to the change of basis. But it is not necessary to be aware of this invariance. A student may think that the minimal polynomial must be obtained from the matrix representation in the standard basis and choose the standard basis to represent T. Representing T relative to B requires a more formal understanding of the matrix representation: the columns are the coordinate vectors of the images of the elements of B relative to B. In calculating the matrix of T relative to the standard basis, one may believe, as many students do, that the columns are, directly, the images (not coordinate vectors of the images) of the vectors of the basis. There is no need to use the notion of coordinate vector, so finding the matrix of T in the standard basis is conceptually more economical.

To find the matrix of T relative to the standard basis, one needs to know $T(\begin{bmatrix}1\\0\end{bmatrix})$ and $T(\begin{bmatrix}0\\1\end{bmatrix})$. There are many ways of getting this information from the given values of T. All ways require knowing that linear transformations preserve linear combinations, although not all require a completely conscious and verbalized such knowledge. One can find a formula for representing any vector as a linear combination of the vectors of the basis B, and then apply it to the vectors of the standard basis. But this is not economical because, first of all, it is not necessary for solving the problem, and, moreover, it is long and one risks making mistakes in the calculations. Time is saved by a process of "systematic guessing".

This is how one of the students appeared to have thought about it. How to get $\begin{bmatrix}1\\0\end{bmatrix}$ from $\begin{bmatrix}1\\2\end{bmatrix}$ and $\begin{bmatrix}3\\4\end{bmatrix}$? First, how to get 0 from 2 and 4? By multiplying 2 by −2 and adding it to 4. By chance $(-2) \cdot 1 + 1 \cdot 3 = 1$.

So $\begin{bmatrix}1\\0\end{bmatrix} = -2\begin{bmatrix}1\\2\end{bmatrix} + \begin{bmatrix}3\\4\end{bmatrix}$.

Now, how to get $\begin{bmatrix}0\\1\end{bmatrix}$ from $\begin{bmatrix}1\\2\end{bmatrix}$ and $\begin{bmatrix}3\\4\end{bmatrix}$? Using a similar technique, we start by asking, how to get 0 from 1 and 3? Of course, one can take, $0 = -3 \cdot 1 + 1 \cdot 3$.

In this case, $-3\begin{bmatrix}1\\2\end{bmatrix} + \begin{bmatrix}3\\4\end{bmatrix} = -2\begin{bmatrix}0\\1\end{bmatrix}$.

This yields the representation $\begin{bmatrix}0\\1\end{bmatrix} = 1.5\begin{bmatrix}1\\2\end{bmatrix} - 0.5\begin{bmatrix}3\\4\end{bmatrix}$.

This student's solution looked as follows:

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = T\left(\begin{bmatrix}3\\4\end{bmatrix}\right) - 2T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) = \begin{bmatrix}1\\8\end{bmatrix} - 2\begin{bmatrix}-1\\0\end{bmatrix} = \begin{bmatrix}3\\8\end{bmatrix}$$

$$T\left(\begin{bmatrix}3\\4\end{bmatrix}\right) - 3T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) = -2T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$$

$$\begin{bmatrix}1\\8\end{bmatrix} - 3\begin{bmatrix}-1\\0\end{bmatrix} = \begin{bmatrix}4\\8\end{bmatrix}$$

$$\begin{bmatrix}4\\8\end{bmatrix} = -2T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$$

$$\begin{bmatrix}-2\\-4\end{bmatrix} = T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$$

$$T = \begin{bmatrix}3 & -2\\8 & -4\end{bmatrix}$$

$$\Delta_T(t) = (3-t)(-4-t) + 16 = -12 + t + t^2 + 16 = t^2 + t + 4$$

$$b^2 - 4ac$$

$$1 - 16$$

no roots

$$\therefore\ m_T(t) = t^2 + t + 4$$

This is a correct solution, obtained quickly using all kinds of conceptual shortcuts, based on practical knowledge of theory, as well as notational shortcuts (e.g. metonymies, such as " $T = \begin{bmatrix}3 & -2\\8 & -4\end{bmatrix}$ ", where the operator is identified with its matrix representation).

In contrast, some students' solutions suggested that they have studied the theory, but not too well, and it has not yet become part of their practical knowledge. They were not able to quickly choose the relevant elements of this theory for solving the problem. Here is an example.

$\Rightarrow$ we have to find $\Delta_T(t)$
to be able to find $m_T(t)$
and take $m_T(t)$ = LCM $\Delta_T(t)$

In the previous quiz, there was a question, "Define the notion: minimal polynomial of a matrix". This student's response was: "least minimal coefficient of a matrix". This sounds quite mysterious, but let's look at the first letters of the three first words in her "definition": "LMC". This looks like "LCM", which appeared in a theorem about the minimal polynomial of a block diagonal matrix: the minimal polynomial of such a matrix is equal to the least common multiple of the minimal polynomials of the blocks. Maybe this is the association she made, and she ended up associating the minimal polynomial with block diagonal matrices. In this previous quiz, there was also a computational exercise: to calculate the characteristic and the minimal polynomials of a block diagonal matrix. She correctly calculated the characteristic polynomials of the blocks, but the notation she used suggested that she was confused, perhaps not only about the notation but also about the nature of the polynomials. She seemed to understand the characteristic polynomial as a function of the matrix but not of a real variable: she wrote, for example, $\Delta(A_1)$ instead of $\Delta_{A1}(t)$. She appeared to know little theory, that's why she could not remember the notation and confused the techniques with the definitions.

Conclusions

Theoretical thinking is, no doubt, necessary to understand even the most basic concepts of linear algebra. But it is not sufficient for solving problems that are not just slightly different versions of worked out textbook examples, and certainly not sufficient for applying or developing linear algebra.

A "practical understanding" of the theory is necessary

to find the smart shortcuts,

the "easy ways" of doing things.

Mathematics is not just about solving problems. It is about finding smart ways of solving problems. Being a thorough, accomplished theoretical thinker doesn't make one a good mathematician, although it may be sufficient for the making of a philosopher.

Transversal categories of thinking such as theoretical/practical thinking do not help the teacher to organize the mathematical content of a course so that students gain a deeper conceptual understanding and both theoretical and practical knowledge of it. Didactic knowledge cannot be derived from general epistemological categories or from psychological studies of cognition. This knowledge requires the study of the mathematical content of teaching. I do not mean that research in mathematics education should return to a *Stoffdidaktik*-like approach to mathematics education. I am not saying that didactic knowledge should be reduced to the study of the mathematical content of teaching. All I am saying is that it should not be replaced by general epistemology or by psycho-social theories of learning and teaching anything.

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