Difficulties in Learning Algebra

Nov. 11, 2008

Overview

1. Evidence of students' difficulties with algebra;
2. Theories about the sources of the difficulties;
3. Attempts to improve the teaching of algebra

Evidence of students' difficulties with algebra

A feature of algebra: the use of symbols, or letters to represent numbers (such as a, x, or y) and expressions (such as $y = 3x + 5, z = x^2 - 3y$). In learning algebra, it is the first difficulty for students.

Evidence of students' difficulties with algebra

A feature of algebra: the use of symbols, or letters to represent numbers (such as a, x, or y) and expressions (such as $y = 3x + 5, z = x^2 - 3y$). In learning algebra, it is the first difficulty for students.

Example 1

Question: What can you write for the perimeter of a polygon with *n* sides altogether, each of length 2?

Michelle: No, · · · Unless *n* stands for ... like say *n*s in the alphabet, somewhere along the numbers... if *n* stands for one of those, then you can say what *n* is.

Teacher: How do you mean?

M: Well, say *n*'s 14.

T: How did you get that?

M: Same as I said before, to get *n* for a number, I got 14, it is 14 along.

T: Oh, did you count along the alphabet?

M: Yes, so that's 14, so I took that for *n*, so it should be ... 28, to make up the perimeter.

T: 28?

M: Yes, you add up all the 2's.

(Booth 1984, p. 33)

Example 2

Question: What can you write for the perimeter of a polygon with n sides altogether, each of length 5?

Marie: n times 5. It's the number of sides times how long each side is, only you don't know how many sides, so all you can do is the n times 5.

Teacher: So the answer is n times 5?

M: Well, you can't give a proper answer, because you don't know what n is. If I knew n, I could work it out, but as it is, all you can put is $5n$.

(Booth 1984, p. 35)

Since letters instead of numbers are used in operations $+, -, \times, \div$, many errors are related to the three laws (commutative, associative and distributive laws), such as

Since letters instead of numbers are used in operations $+, -, \times, \div$, many errors are related to the three laws (commutative, associative and distributive laws), such as

Example 3

$$a(b+c) = ab + c;$$

$$\frac{ab+c}{a} = b + c.$$

Since letters instead of numbers are used in operations $+, -, \times, \div$, many errors are related to the three laws (commutative, associative and distributive laws), such as

Example 3

$$a(b+c) = ab + c;$$

$$\frac{ab+c}{a} = b + c.$$

Example 4, Errors related to "$=$":

$$2a + 5b = 7ab$$

(MATH 208 class at Concordia)

Teacher: Since $(a-b)(a+b) = a^2 - b^2$, $a^2 - b^2 = (a-b)(a+b)$.

Students: Why?

Theories about the sources of the difficulties

"Algebra is a language for describing actions on, and relationship among, quantities. As with any language difficulties may arise from feature of the language itself or in translating from one language to another. Within the language of algebra, most linguistic difficulties are related to variables and expressions; most translation difficulties arise in translating word problems into equations."

(S. Wagner and S. Parker, (1999),*Advancing Algebra in B. Moses (Ed), Algebraic Thinking, Reston, VA. NCTM. INC. p328)*

Theories about the sources of the difficulties

"Algebra is a language for describing actions on, and relationship among, quantities. As with any language difficulties may arise from feature of the language itself or in translating from one language to another. Within the language of algebra, most linguistic difficulties are related to variables and expressions; most translation difficulties arise in translating word problems into equations."

(S. Wagner and S. Parker, (1999),*Advancing Algebra in B. Moses (Ed), Algebraic Thinking, Reston, VA. NCTM. INC. p328)*

Difficulties usually arise at the turning points of the knowledge.

Ex 1 and 2: (linguistic difficulties)

We know that letters also appear in arithmetic, but in very different way. For example in arithmetic, we may use letter a to denote "apples" and p to denote "pears", but in algebra, we normally use a to represent the number of apples, and p, the number of pears. Based on their old knowledge in arithmetic, students think that letters are names for concrete things. So for them, it is easier to think that $3a$ does not mean 3 times the number of apples, but 3 apples.

The confusion here is also due to the omission of the operation sign for multiplication ($3a$ (or $3 \cdot a$) for $3 \times a$).

“Because we use two distinct symbol systems (letters and numbers) together in algebra, and because these systems follow different rules, we gain some us some economy of the notation, but at the expense of possible confusion.”

(S. Wagner and S. Parker, (1999),*Advancing Algebra in B. Moses (Ed), Algebraic Thinking, Reston, VA. NCTM. INC. p331)*

In Ex. 1, the student does not understand that n is an abstract number and it can denote any number. She gives a value to n which makes sense to her.

Ex3:

Although, in arithmetic we also have the commutative, associative and distributive laws in operations $+, -, \times, \div$, when we calculate $3(2+7)$, we usually do it as

$$3(2+7) = 3 \times 9 = 27,$$

Ex3:

Although, in arithmetic we also have the commutative, associative and distributive laws in operations $+, -, \times, \div$, when we calculate $3(2+7)$, we usually do it as

$$3(2+7) = 3 \times 9 = 27,$$

rather than

$$3(2+7) = 3 \times 2 + 3 \times 7 = 6 + 21 = 27.$$

So there is much less opportunity for using the distributive law. But for $3(a+6)$, one has to use this law:

$$3(a+b) = 3a + 3b.$$

If he/she does not know the law well, then errors can happen.

Ex4.

The reasons for the errors related to "=":
For some students. "=" means to write down the answer, that it, " = " means "gives", an unidirectional symbol proceeding an answer, like $2 + 6 = 8$.
The expressions $8 = 2 + 6$ and $3 + 5 = 2 + 6$ don't make sense to them.

Similarly, $(a - b)(a + b) = a^2 - b^2$ means $(a - b)(a + b) \Rightarrow a^2 - b^2$.

Attempts to improve the teaching of algebra

Since the errors happen most frequently at the turning points of the knowledge, when teaching new knowledge, we should emphasize its relationship to and difference with the old knowledge, to show students how it is derived from the old knowledge and what is the development.

Attempts to improve the teaching of algebra

Since the errors happen most frequently at the turning points of the knowledge, when teaching new knowledge, we should emphasize its relationship to and difference with the old knowledge, to show students how it is derived from the old knowledge and what is the development.

Example 5

Teaching Exponentiation: Extension of exponents from positive exponents to 0 and negative exponents.
When n is a positive integer, exponentiation corresponds to repeated multiplication:

$$a^n = \underbrace{a \times a \times \cdots \times a}_{n}, \quad a > 0, n \in \mathbb{N}, \tag{1}$$

which satisfies the laws of exponents:

$$a^m a^n = a^{m+n}, \quad (m > 0, n > 0) \tag{2}$$

$$\frac{a^m}{a^n} = a^{m-n}, \quad (m > n > 0) \tag{3}$$

and

$$(a^m)^n = a^{mn}. \tag{4}$$

The zero exponent

$$a^0 = 1 \tag{5}$$

has no longer the same meaning as a^n in (1).

Explanation of (5):
Extend the law $\frac{a^m}{a^n} = a^{m-n}$ to the case $n = m$, the right-hand side becomes a^0 (undefined in the original sense) while the left-hand side is $\frac{a^m}{a^m} = 1$.

Here the new knowledge is obtained by **defining**, the positive exponent is extended to 0. $a^0 = 1$ is a definition, but there are some good reason for this definition.
Principle of mathematical "ethic": Old laws should be the special case of new laws and new knowledge is consistent to the old one.

Negative integer exponents

Analogously, if $n > m$, say, $n = m + 1$ in (3), the equation would read

$$\frac{1}{a} = a^{-1}.$$

So we define $a^{-1} = \frac{1}{a}$. , the exponent is extended to -1, a definition again.
Then for $n > 0$,

$$a^{-n} = (a^{-1})^n = \frac{1}{a^n},$$

extended to any negative exponent by reasoning.

About "="

To help students develop the algebraic concept of "=", we can show students that both sides are equivalent to each other, that is,

"$a = b$" means "a gives b " and " b gives a".

Recall Ex. 4, $(a-b)(a+b) \Leftarrow a^2 - b^2$:

$$a^2-b^2 = a^2-ab+ab-b^2 = a(a-b)+b(a-b) = (a-b)(a+b).$$

Also

$$8 = 8 - 6 + 6 = 2 + 6.$$